Multi-Leader-Follower Games: non cooperative and hierarchical/bilevel interactions

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Optimization Workshop - March 24th, 2023

- Professor in Applied Mathematics at Univ. of Perpignan

Perpignan, France

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    - Demand-side management
    - and others....(management of renewable energy plants)

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- Research lab.: PROMES (CNRS)

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Optimization /Math. programming

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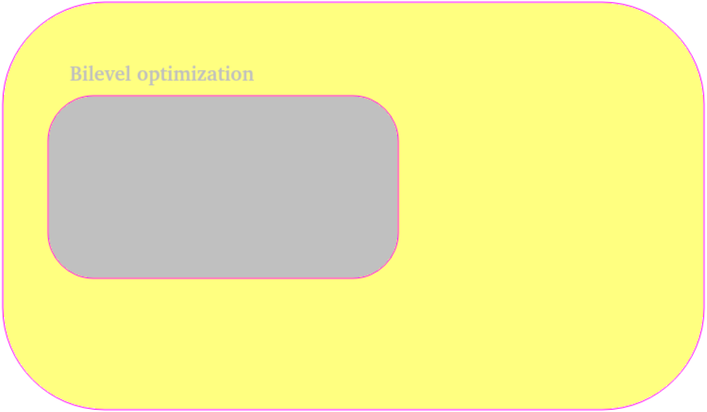

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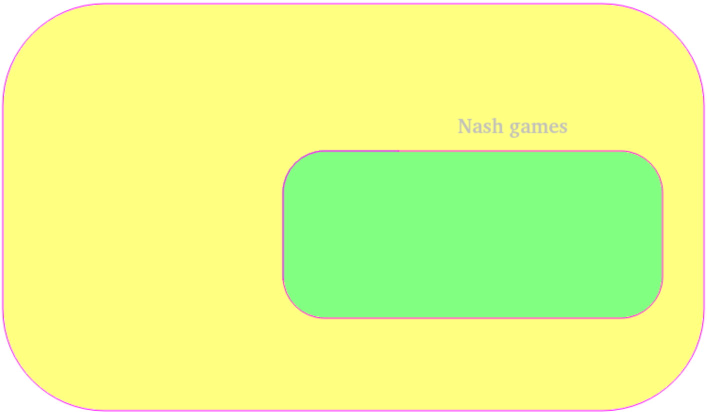

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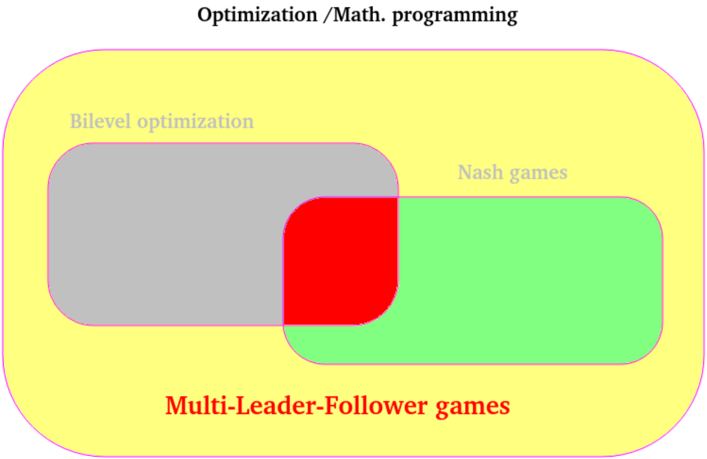
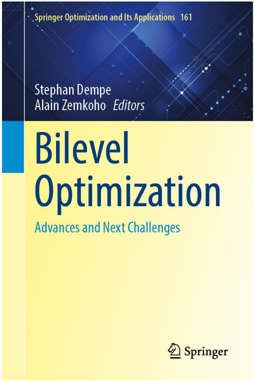


Hell zone!!!!!

An advertisement

A short state of art on Multi-Leader-Follower games, D.A. and A. Svensson, in a book dedicated to Stackelberg, editors A. Zemkoho and S. Dempe, Springer Ed. (2021)



Menu (for the three days...)

- Lecture 1: Definitions, motivations and well-posedness
- Lecture 2: Some first applications
- Lecture 3: Reformulations (differentiable and non differentiable cases)

Lecture 1

- *Definitions of different MLMFG*

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- *Definitions of different MLMFG*
- *Is this well-posed? Is it meaningfull?*

Generalized Nash game (GNEP):

$$\boxed{\begin{array}{ll} \min\limits_{x_1} & \theta_1(x_1, x_{-1}) \\ \text{s.t.} & \{\ x_1 \in K_1(x_{-1}) \end{array}} \quad \ldots \quad \boxed{\begin{array}{ll} \min\limits_{x_n} & \theta_n(x_n, x_{-n}) \\ \text{s.t.} & \{\ x_n \in K_n(x_{-n}) \end{array}}$$

Generalized Nash game (GNEP):

$$\begin{array}{ll} \min_{x_1} & \theta_1(x_1, x_{-1}) \\ \text{s.t.} & \{ \; x_1 \in K_1(x_{-1}) \end{array} \quad \cdots \quad \begin{array}{ll} \min_{x_n} & \theta_n(x_n, x_{-n}) \\ \text{s.t.} & \{ \; x_n \in K_n(x_{-n}) \end{array}$$

So we have n players and they are interacting in a **non cooperative way** (join venture is forbidden or impossible...)

$\bar{x}$ is a Generalized Nash Equilibrium
if and only if
in case a player i would decide to unilaterally deviate from $\bar{x}_i$
(say to $\tilde{x}_i$) then

"he will loose" := "$\theta_i(\tilde{x}_i, \bar{x}_{-i}) \geq \theta_i(\bar{x}_i, \bar{x}_{-i})$" !!!!

Motivation for GNEP

Power allocation in telecommunication

Consider a DSL network (Digital Subscriber Line)

- DSL customers connected to the central by dedicaced lines
- wires are bundled together in telephone cables
- electromagnetic coupling $\Rightarrow$ degradation of quality

<u>control variables:</u>
for each wire q and each subcarrier k, p_k^q= power allocated for transmission
<u>constraints:</u>
for each wire q: maximum achievable transmission rate R_q (transmission quality)
It depends of $(p_k)_{k=1,N}$ (power allocations across available subcarriers for q) and of $p^{-q} = (p^r)_{r \neq q}$ (strategies of the other wires)

Control variables:

$$R_q(p^q, p^{-q}) = \sum_{k=1}^{N} log(1 + sinr_k^q)$$

where

$$sinr_k^q = \frac{|H_k^{qq}|^2.p_k}{\sigma_q^2}(k) + \sum_{r \neq q} |H_k^{qr}|.p_k^r.$$

(Signal-t-Interference Noise Ratio)
Garantee of minimal transmission rate $R_q(p^q, p^{-q}) \geq R_q^*$.

Model:
each wire wants is a player of the game whose objective is to minimize to total power used for transmission, with the constraint that the maximum transmission rate is at least R_q^*, that is

$$\text{for any } q, \quad \text{solve } (P_q) \quad \begin{array}{ll} \min & \sum_k = 1^k p_k^q \\ s.t. & \left\{ \begin{array}{l} R_q(p^q, p^{-q}) \geq R_q^* \\ p_k^q \geq 0 \end{array} \right. \end{array}$$

Theorem

Assume that

- *for every player* ν*, the loss function* θ_ν *is continuous on* $\mathbb{R}^n$ *and semistrictly quasiconvex with respect to the* ν*-th variable;*
- *the set* X *is nonempty, convex and compact.*

Then the Nash equilibrium problem admits a solution.

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Theorem (Debreu)

Assume that

- *for every player* ν*, the loss function* θ_ν *is continuous on* $\mathbb{R}^n$ *and quasiconvex with respect to the* ν*-th variable;*
- *the set-valued map is continuous with nonempty convex compact values*

. Then the generalized Nash equilibrium problem admits a solution.

Bilevel optimisation

Bilevel problem:

$$
\boxed{
\begin{array}{ll}
\text{"}\min_x\text{"} & \theta(x,y) \\
\text{s.t.} & \left\{ \begin{array}{l} x \in X(y) \\ \min_y \quad \phi(x,y) \\ \ \text{s.t.} \quad\ \ y \in Y(x) \end{array} \right.
\end{array}
}
$$

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- Ambiguities...

But also...

Bilevel problem:

$$
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\end{array}
$$

- Ambiguities...
- Leader or Follower points of view?

BL = Single-Leader-Single-Follower game

A Bilevel Problem consists in an **upper-level/leader's problemthis**

$$\begin{array}{rl} \text{“}\min_{x\in\mathbb{R}^n}\text{”} & F(x,y) \\ s.t. & \begin{cases} x \in X(y) \\ y \in S(x) \end{cases} \end{array}$$

where $\emptyset \neq X \subset \mathbb{R}^n$ and $S(x)$ stands for the solution set of its **lower-level/follower's problem**

$$\begin{array}{rl} \min_{y\in\mathbb{R}^m} & f(x,y) \\ s.t & g(x,y) \leq 0 \end{array}$$

A trivial example

Consider the following simple bilevel problem

$$
\begin{array}{ll}
\text{“}\min_{x\in\mathbb{R}}\text{”} & x \\
s.t. & \left\{ \begin{array}{l} x \in [-1,1] \\ y \in S(x) \end{array} \right.
\end{array}
$$

with $S(x) =$ “y solving

$$
\begin{array}{ll}
\min_{y\in\mathbb{R}} & -xy \\
s.t & x^2(y^2-1) \leq 0 \text{ ”}
\end{array}
$$

A trivial example

Lower level problem:

$$\begin{array}{cc} \min_{y\in\mathbb{R}} & -x.y \\ s.t & x^2(y^2-1)\leq 0 \text{ ''} \end{array}$$

Note that the solution map of this convex problem is

$$S(x) := \begin{cases} \{1\} & x<0 \\ \{-1\} & x>0 \\ \mathbb{R} & x=0 \end{cases}$$

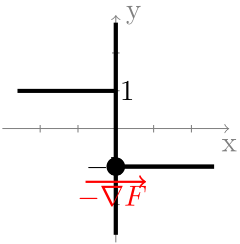
Thus for each $x \neq 0$ there is a unique associated solution of the lower level problem

A trivial example

Lower level problem:

$$\min_{y\in\mathbb{R}} \quad -xy$$
$$s.t \quad x^2(y^2-1)\leq 0 \text{ ''}$$

Note that the solution map of this convex problem is



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Ambiguity: Optimistic approach

An *Optimistic Bilevel Problem* consists in an
upper-level/leader's problem

$$\min_{x\in\mathbb{R}^n} \color{red}{\min}_{y\in\mathbb{R}^m} \quad F(x,y) \qquad s.t. \quad \begin{cases} x \in X(y) \\ y \in S(x) \end{cases}$$

where $\emptyset \neq X \subset \mathbb{R}^n$ and $S(x)$ stands for the solution set of its
lower-level/follower's problem

$$\min_{y\in\mathbb{R}^m} \quad f(x,y) \qquad s.t \quad g(x,y) \leq 0$$

Ambiguity: Pessimistic approach

An *Pessimistic Bilevel Problem* consists in an
upper-level/leader's problem

$$\begin{array}{rl} \min_{x\in\mathbb{R}^n} \max_{y\in\mathbb{R}^m} & F(x,y) \\ s.t. & \begin{cases} x \in X(y) \\ y \in S(x) \end{cases} \end{array}$$

where $\emptyset \neq X \subset \mathbb{R}^n$ and $S(x)$ stands for the solution set of its
lower-level/follower's problem

$$\begin{array}{rl} \min_{y\in\mathbb{R}^m} & f(x,y) \\ s.t & g(x,y) \leq 0 \end{array}$$

Ambiguity: the most simple

And of course the "confortable situation" corresponds to the case of a unique response

$$\forall\, x \in X, \quad S(x) = \{y(x)\}.$$

Then

$$\begin{array}{rl} \min_{x\in\mathbb{R}^n} & F(x, y(x)) \\ s.t. & \left\{\ x \in X(y(x))\right. \end{array}$$

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For example when

for any x, $g(x,\cdot)$ *is quasiconvex and* $f(x,\cdot)$ *is strictly convex.*

Ambiguity: Selection approach

An *"Selection-type" Bilevel Problem* consists in an **upper-level/leader's problem**

$$\min_{x\in\mathbb{R}^n} \quad F(x, y(x))$$
$$s.t. \quad \begin{cases} x \in X(y(x)) \\ y(x) \textit{ is a uniquely determined selection of } S(x) \end{cases}$$

J. Escobar & A. JofrÃ©, *Equilibrium Analysis of Electricity Auctions* (2011)

Ambiguity: The new probabilistic approach

Recently, D.Salas and A. Svensson proposed a **probabilistic approach**:

- *Consider a probability on the different possible follower's reactions*
- *Minimize the expectation of the leader(s)*

An alternative point of view

Instead of considering the previous (optimistic) formulation of BL:

$$\begin{array}{rl} \min_{x\in\mathbb{R}^n} \min_{y\in\mathbb{R}^m} & F(x,y) \\ s.t. & \left\{ \begin{array}{l} x \in X \\ y \in S(x) \end{array} \right. \end{array}$$

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one can define the (optimistic) value function

$$\varphi_{min}(x) = \min_y \{F(x,y) : g(x,y) \leq 0\} \tag{1}$$

and the BL problem becomes

$$\begin{array}{rl} \min_{x\in\mathbb{R}^n} & \varphi_{min}(x) \\ s.t. & x \in X \end{array}$$

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one can define the (pessimistic) value function

$$\varphi_{max}(x) = \max_y \{F(x,y) : g(x,y) \leq 0\} \tag{2}$$

and the Bl problem becomes

$$\begin{array}{rl} \min_{x\in\mathbb{R}^n} & \varphi_{max}(x) \\ s.t. & x \in X \end{array}$$

This is the point of view presented in Stephan Dempe's book:

$$\min_{x\in\mathbb{R}^n} \min / \max_{y\in\mathbb{R}^m} \quad F(x,y) \quad s.t. \quad \begin{cases} x \in X \\ y \in S(x) \end{cases}$$

vs

$$\min_{x\in\mathbb{R}^n} \quad \varphi_{min/max}(x) \quad s.t. \quad x \in X$$

This is the point of view presented in Stephan Dempe's book:

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It immediately raises the question

What is a solution??

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It immediately raises the question

What is a solution??

- *an optimal x = leader's optimal strategy?*
- *an optimal couple (x, y) = couple of strategies of leader and follower?*