Discrete random variables — Expectation and variance

MTH382 Probability Theory for Finance and Actuarial Science

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Overview

This lecture is bears the most important tools in probability, which are expectation and variance of a r.v. Expectation could be thought of as the *mean* (or *average*) of all possible values produced by a random variable.

Expectation

Definition

Always consider a probability space $(\Omega, \mathcal{F}, P)$.

Definition 1.
Let X be a discrete r.v. with values in $E \subset \mathbb{R}$ with the condition that

$$\text{either} \quad E \subset \mathbb{R}_+ \quad \text{or} \quad \sum_{x \in E} |x| \, P(X = x) < \infty. \tag{1}$$

Then the **expectation** (or **expected value**) of X, denoted with $\mathbb{E}X$ (or $\mathbb{E}[X]$), is defined by

$$\mathbb{E}X := \sum_{x \in E} x P(X = x).$$

▲

Remark.
The condition (1) is only to gurantee that the summation in the definition of $\mathbb{E}X$ is well-defined. Note that an expectation could takes an infinite value.

Examples

We begin with the simplest examples.

Example 2.
Compute $\mathbb{E}X$ where X represents the following situtations.

(1) X is the result of tossing a fair dice.

(2) X is the result of tossing an unfair dice with

$$P(\text{⚀}) = 0.2, \quad P(\text{⚁}) = 0.1, \quad P(\text{⚂}) = 0.2, \quad P(\text{⚃}) = 0.1, \quad P(\text{⚄}) = 0.3, \quad P(\text{⚅}) = 0.1.$$

Example

Now we move to a more complicate example.

Example 3.
Consider S_n which is the number of heads appearing in tossing a coin n times.
Calculate the expected value $\mathbb{E}[S_n]$.

Example

The next example shows that even a r.v. X has finite values, its expectation could be infinite.

Example 4.
Let X be a r.v. taking values in $E = \mathbb{N}$ with the distribution

$$P(X = n) = \frac{1}{cn^2}$$

for each $n \in \mathbb{N}$, where $c = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$.

(a) Show that P is a probability measure.

(b) Show that $\mathbb{E}X = +\infty$

Expectation

Properties of $\mathbb{E}$

Properties of $\mathbb{E}$

Theorem 5.
The following properties hold.

(a) *The expectation is linear, i.e.*

$$\mathbb{E}[\lambda_1 X_1 + \cdots + \lambda_n X_n] = \lambda_1 \mathbb{E} X_1 + \ldots \lambda_n \mathbb{E} X_n$$

for any $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$*, where* $X_1, \ldots, X_n : \Omega \to \mathbb{R}$ *are discrete r.v.s.*

(b) *The expectation is monotone, i.e. for any discrete r.v.s* $X_1, X_2 : \Omega \to \mathbb{R}$ *such that* $X_1 \leq X_2$*, then*

$$\mathbb{E} X_1 \leq \mathbb{E} X_2.$$

(c) *The expectation satisfies the triangle inequality, i.e.*

$$|\mathbb{E} X| \leq \mathbb{E}\,|X|$$

Variance

Definition

Take a discrete r.v. X taking values in $E \subset \mathbb{R}$. We say that it is **integrable** if $\mathbb{E}[|X|] < \infty$ and **square integrable** if $\mathbb{E}[X^2] < \infty$.

Definition 6.
Let X be a discrete r.v. with values in $E \subset \mathbb{R}$ which is square integrable, and let $\mu := \mathbb{E}X$. The **variance** of X is defined by

$$\mathrm{Var}[X] := \mathbb{E}[(X-\mu)^2] = \sum_{x \in E} (x-\mu)^2 P(X = x).$$

A lot of times, we use the notation $\sigma^2 := \mathrm{Var}[X]$.

The variance $\mathrm{Var}[X]$ describes how much the values of X could be *away* from its average μ.

A better formula

Proposition 7.
The variance also has the following expression

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}X)^2,$$

or in other words,

$$\sigma^2 = \mathbb{E}[X^2] - \mu^2.$$

Example 8.
Consider two r.v.s X and Y, where we express them with their distribution functions

$$\pi_X(x) := \begin{cases} 0.5 & \text{if } x = 10, \\ 0.5 & \text{if } x = -10, \\ 0 & \text{otherwise,} \end{cases} \qquad \pi_Y(y) := \begin{cases} 1 & \text{if } y = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Compare and explain between their expectations $\mathbb{E}X$ and $\mathbb{E}Y$ and also their variances σ_X^2 and σ_Y^2.

Example 9.
Compute the variances of r.v.s in the previous examples.

Variance

Propoerties of Var

Properties of Var

Theorem 10.
Let X be a discrete r.v. and $a, b \in \mathbb{R}$, then

$$\operatorname{Var}[aX + b] = a^2 \operatorname{Var}[X].$$

Theorem 11.
If $X_1, \ldots, X_n$ are independent discrete r.v.s, then

$$\operatorname{Var}[X_1 + \cdots + X_n] = \operatorname{Var}[X_1] + \cdots + \operatorname{Var}[X_n].$$

Example 12.
Consider repeatedly tossing a weighted coin with $P(\mathrm{H}) = p$ and $P(\mathrm{T}) = 1 - p$, with $p \in (0, 1)$. Find the expected value and variance of S_n representing the number of heads appearing in the n tosses.

Moments

Moments

We have studied $\mathbb{E}X$ and $\mathsf{Var}[X]$, which is related to $\mathbb{E}[X^2]$. One might be curious about $\mathbb{E}[X^k]$ for larger $k \in \mathbb{N}$ and in fact, some of them has special meanings. For examples, $\mathbb{E}[X^3]$ is the **skewness** (the lack of symmetry) and $\mathbb{E}[X^4]$ is the **kurtosis** (the fatness of the tail).

In fact, $\mathbb{E}[X^k]$ is called the k^{th} **moment** of X.

One could also consider the **moment generating function** $M_X(s) := \mathbb{E}[e^{sX}]$, since this function captures *all* the moments of X from the fact that

$$\left.\frac{d^k M_X}{ds^k}\right|_{s=0} = \mathbb{E}[X^k]$$

for all $k \in \mathbb{N}$.

Example 13.
Consider the case of a single coin tossing with a r.v. X being 1 when H occurs and 0 when T occurs. Then

$$\pi_X(x) = \begin{cases} 0.5, & \text{if } x = 0, 1, \\ 0, & \text{otherwise.} \end{cases}$$

- Calculate the moment generating function M_X.
- Show that $M'_X = \mathbb{E}X$ and $M''_X = \mathbb{E}[X^2]$.

Example

Example 14.
Suppose that X is a discrete r.v. whose moment generating function is

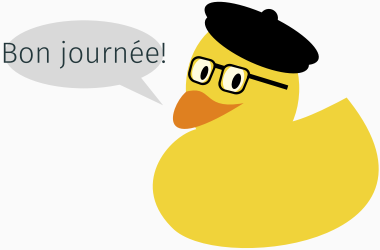
$$M_X(s) = \frac{1}{3}(1 + e^s + e^{2s}).$$

What is the expectation and variance of X ?

Takeaways

Takeaways

- $\mathbb{E}$ is the **mean** and Var is the **variance** of the data encoded by a given random variable.
- $\mathbb{E}$ is linear but Var is not.
- Var is additive only for independent random variables.
- The moment generating function is capable of describes all the moments of any r.v. and it could be useful in computing the variance.


Bon journée!