Discrete random variables — Independent and identically distributed sequences

MTH382 Probability Theory for Finance and Actuarial Science

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Overview

In this lecture, we consider a multiple variables and discuss their independence. The ultimate goal here is to be able to investigate a sequence of independent and identically distributed sequence.

Independent random variables

Definition

Always consider a probability space $(\Omega, \mathcal{F}, P)$.

Definition 1.
Let E_1 and E_2 are two countable sets. Two discrete r.v.s $X_1 : \Omega \to E_1$ and $X_2 : \Omega \to E_2$ are called **independent** if

$$P(X_1 = x_1,\ X_2 = x_2) := P(\{X_1 = x_1\} \cap \{X_2 = x_2\}) = P(X_1 = x_1)P(X_2 = x_2),$$

for all $x_1 \in E_1$ and all $x_2 \in E_2$.

This concept of independence generalizes naturally to any finite case.

Definition 2.
Let $(\Omega, \mathcal{F})$ be a measurable space, $E_1, \ldots, E_n$ are two countable sets. For each $i = 1, \ldots, n$, let $X_i : \Omega \to E_i$ be r.v.s. We say that $X_1, \ldots, X_n$ are **independent** if

$$P(X_1 = x_1, \ldots, X_n = x_n) = P(X_1 = x_1) \ldots P(X_n = x_n),$$

for all $x_i \in X_i$ and $i = 1, \ldots, n$.

Example

Consider the switching network shown in Figure 1.1, where there are two switches, s_1 and s_2. Each switch has two states, namely, Working and Failed. The switches s_1 and s_2 have failure rates of 0.02 and 0.05, respectively.

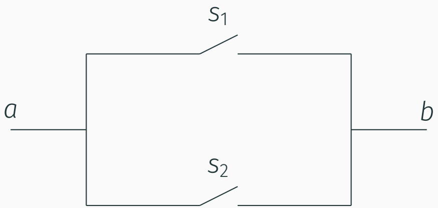


Figure 1.1: Switching network with two switches.

In order to have the electricity flows between points a and b, at least one of the two switches must be in the Working state. What is the probability that no electricity flows between a and b ?

Example

Consider the switching network shown in Figure 1.2, where there are four switches, s_1, s_2, s_3, s_4. Each switch can be in one of the two positions, On or Off, with a probability of $\frac{1}{2}$ of being in the On position.

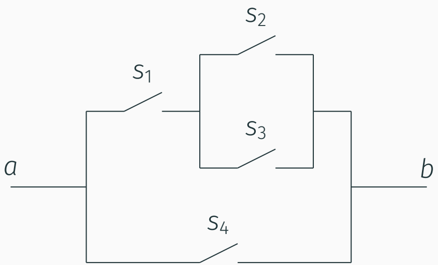


Figure 1.2: Switching network with four switches.

What is the probability that electricity flows between a and b ?

Independent and identically distributed sequence

Definition 3.
A sequence (X_n) of discrete r.v.s is said to be **independent** if any finite collection $X_{i_1}, \cdots, X_{i_r}$ extracted from this sequence are independent.

Definition 4.
If X is a discrete r.v., then its **distribution function** is $\pi_X : X \to [0,1]$ with

$$\pi_X(x) := P(X = x).$$

This π_X is also called the **probability law** or simply the **law** of X. It represents the induced probability on the image of X.

Definition 5.
A sequence (X_n) of discreate r.v.s is said to be **independent and identically distributed** (briefly **i.i.d.**) if

- all X_n's take their values in the same set E,
- the sequence (X_n) is independent as of Definition 3,
- the probability distribution functions of all X_n's are the same.

Example

Natural examples of i.i.d. sequences are those related to infinite random experiments.

Example 6.
Consider tossing a single coin repeatedly and indefinitely. If X_n is the r.v. representing the outcome of the n^{th} toss, prove that the sequence (X_n) is i.i.d.

Example 7.

Consider again an indefinite coin tossing where (X_n) is an i.i.d. sequence of r.v. and for each $n \in \mathbb{N}$, X_n represents the outcome of the n^{th} toss (0 representing T and 1 representing H). Now, for each $n \in \mathbb{N}$, we consider an r.v. $S_n : \Omega \to \mathbb{R}$ defined by

$$S_n = X_1 + \cdots + X_n.$$

This r.v. is the number of heads appearing during the first n tosses.

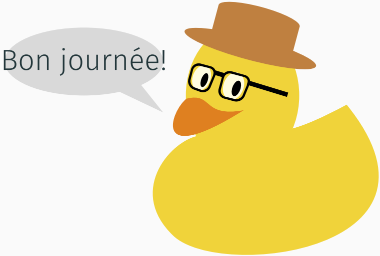
(a) Show that (S_n) is not i.i.d.

(b) Compute the probability of the event $S_n = k$ for $k = \{1, \ldots, n\}$.

Takeaways

Takeaways

- Independence of random variables is stronger than that of events — It includes independence of all the events caused by different values of the involved random variables.
- An i.i.d. sequence is heavily used in **stochastic processes** that describes a sequence of random experiments or sequence of events that occur over time.


Bon journée!