Conditional probability and Independent events

MTH382 Probability Theory for Finance and Actuarial Science

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Overview

The aim of this lecture is to understand **conditional probability**, the concept that describes a probability of an event knowing that an event occured.

The key theorem of this topic is the **Bayes' rule**, which is used largely in modern statistics and data science. It captures the way a probability is updated with new set of information.

Conditional probability

Motivating example

Consider a disease that 0.2% of Thai population is infected. This means the probability that a person has this disease is $P(\text{disease}) = 0.002$.

On the other hand, if a person has a test and the result is positive then the probability of this person having the disease is $P(\text{disease} \mid \text{positive}) = 0.94$.

The latter one, which reflects the accuracy rate of the test, is referred to the **conditional probability** of being infected given that the test result is positive.

This means a probability of an event is changed if some knowledge is provided.

Always consider $(\Omega, \mathcal{F}, P)$ be a probability space.

Definition 1.
Let $A, B \in \mathcal{F}$ be two events with $P(B) > 0$. The **conditional probability of** A **given** B is the quantity

$$P(A \mid B) := \frac{P(A \cap B)}{P(B)}.$$

▲

Examples

Example 2.
Consider tossing a coing three times. Find the probabilites of ...

(a) getting all heads,
(b) getting all heads, knowing the first toss is head,
(c) getting all heads, knowing the first two tosses are head.

Example 3.
A survey group consists of 1000 people. Among them, 45% are male. Of all the people in this group, 120 males and 180 females used to have COVID-19. What is the probability that an infected member is a female ?

Bayes’ rule

Bayes' rule

Theorem 4 (Bayes' rule of retrodiction).
Let A and B be two events with $P(A), P(B) > 0$. Then

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A)}.$$

Updating probability with information

This theorem is widely used in modern statistics, data science and machine learning. It states a way to *update* a probability of an event B (*prior*) after receiving a new information that an event A occurs with a known *likelihood* $P(A \mid B)$ and *marginal* $P(A)$. The updated probability is known as the *posterior* probability:

$$\underbrace{P(B \mid A)}_{\text{posterior}} = \frac{\overbrace{P(A \mid B)}^{\text{likelihood}}}{\underbrace{P(A)}_{\text{marginal}}} \cdot \underbrace{P(B)}_{\text{prior}}.$$

Example 5.
Suppose that a population is affected by a disease. Suppose that a probability of a person to be infected is known to be $P(\text{Infected}) = 0.3$. The probability of a person to have a positive test result (including both true and false positives) from an self-test kit is $P(\text{Positive}) = 0.32$, while the self-test kit is known to be 85% accurate on infected patients *i.e.* $P(\text{Positive} \mid \text{Infected}) = 0.9$.

Find $P(\text{Infected} \mid \text{Positive})$

Independent events

Definition 6.
Let $(\Omega, \mathfrak{F}, P)$ be a probability space. Then the two events $A, B \in \mathfrak{F}$ are called **independent** if

$$P(A \cap B) = P(A)P(B).$$

▲

Example 7.

- In tossing a coin twice, the events that the first toss is head, and that the second toss is head, are independent.
- In rolling a dice, the events that an outcome is odd, and that an outcome is even, are not independent.

Example 8 (SOA Exam P Sample Question).
An actuary studying the insurance preferences of automobile owners makes the following conclusions:

(i) An automobile owner is twice as likely to purchase collision coverage as disability coverage.
(ii) The event that an automobile owner purchases collision coverage is independent of the event that he or she purchases disability coverage.
(iii) The probability that an automobile owner purchases both collision and disability coverages is 0.15.

Calculate the probability that an automobile owner purchases neither collision nor disability coverage.

Interpretation in term of conditioning

The term **independence** becomes clearer with the following result.

Proposition 9.
Let A and B be two events with $P(B) > 0$. Then A and B are independent if and only if $P(A|B) = P(A)$.

Bayes' rule for partitions and Scenario trees

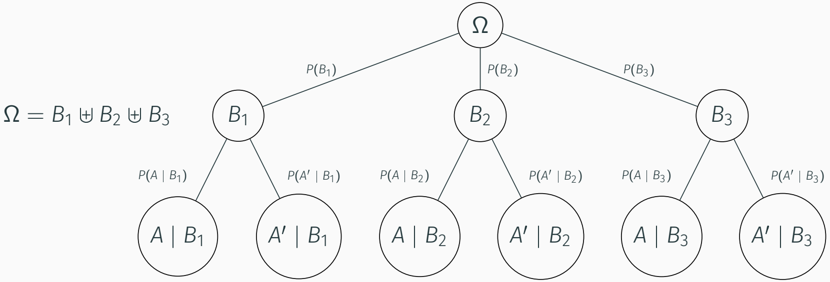
Bayes' rule for partitions

The Bayes' rule for partitions is responsible for the probability calculation using a scenario tree. It is stated as follows.

Theorem 10 (Bayes' rule for partitions).
Let $\{B_i\}_{i\in\mathbb{N}}$ be a family of events that forms a partition of Ω, i.e. the collection $\{B_i\}_{i\in\mathbb{N}}$ is pairwise disjoint and $\Omega = \biguplus_{i=1}^{\infty} B_i$. Suppose that $P(B_i) > 0$ for all $i \in \mathbb{N}$. The the following formula holds

$$P(A) = \sum_{i=1}^{\infty} P(A|B_i)P(B_i)$$

for any event A.



$$P(A) = \quad P(A \mid B_1)P(B_1) \quad + \quad P(A \mid B_2)P(B_2) \quad + \quad P(A \mid B_2)P(B_2)$$

We may draw a scenario tree in such a way that Ω is partitioned into $B_1, \ldots, B_n$, and from here further into subevents $A \mid B_i$ at each node B_i. The above shows an example of a scenario tree with $n = 3$.

Example 11.
Make a calculation, by using the Bayes' rule for partitions, of the probability of having two tails in two consecutive coin tosses. Also link it with the corresponding scenario tree.

Example 12 (SOA Exam P Sample Quesiton).
An auto insurance company isures drivers of all ages. An actuary compiled the following statistics on the company's insured drivers:

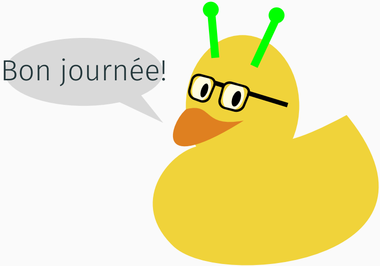
Age of Driver	Probability of Accident	Portion of Company's Insured Drivers
16–20	0.06	0.08
21–30	0.03	0.15
31–65	0.02	0.49
66–99	0.04	0.28

A randomly selected driver that the company insures has an accident. Calculate the probability that the driver was age 16–20.

Takeaways

Takeaways

- Bayes' rule: Updating probability.
- Independence: No updates.
- Bayes' rule for partitions: Scenario trees.
  Breaking a probability down into subcases.


Bon journée!