A formal introduction to probability spaces

MTH382 Probability Theory for Finance and Actuarial Science

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Probability measures

Definition 1.
Let $(\Omega, \mathcal{F})$ be a measurable space. A **probability measure** (or simply a **probability**) on $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \to [0, 1]$ such that the following conditions are satisfied:

(P1) $P(\Omega) = 1$.

(P2) For any sequence of events $(A_i)_{i \in \mathbb{N}}$ in $\mathcal{E}$ such that $A_i \cap A_j = \varnothing$ for all $i, j \in \mathbb{N}$ with $i \neq j$, it holds

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

Whenever P is a probability on $(\Omega, \mathcal{F})$, the tripple $(\Omega, \mathcal{F}, P)$ is called a **probability space**.

Example 2 (SOA Exam P Sample Question).
You are given that $P(A \cup B) = 0.7$ and $P(A \cup B^{\complement}) = 0.9$.
Calculate $P(A)$.

Example 3 (SOA Exam P Sample Question).
In modeling the number of claims filed by an individual under an automobile policy during a three-year period, an actuary makes the simplifying assumption that for all integers $n \geq 0$, $p(n+1) = 0.2p(n)$ where $p(n)$ represents the probability that the policyholder files n claims during the period.
Calculate the probability that a policyholder files more than one claim.

Borel's construction (1D case)

In the unit interval $[0, 1]$, we put $\mathcal{I} := \{(a, b) \mid 0 \leq a < b \leq 1\}$ to be the set of all open subintervals of $[0, 1]$.

Consider $\Omega = [0, 1]$. The Borel σ-field, denoted by $\mathcal{B}$ (or $\mathcal{B}([0, 1])$), is defined by

$$\mathcal{B} = \sigma(\mathcal{I}).$$

Then there exists a unique probability measure $P : \mathcal{B} \to [0, 1]$ such that

$$P(I) = b - a, .$$

for any subinterval $I \subset [0, 1]$ with endpoints $a \leq b$.

A similar result can be carried out in the n-dimensional case.
Consider $\Omega = [0,1]^n$. The **Borel σ-field** over $[0,1]^n = \underbrace{[0,1] \times \cdots \times [0,1]}_{n \text{ times}}$, denoted $\mathcal{B}$ (or $\mathcal{B}([0,1]^n)$), is defined by

$$\mathcal{B}([0,1]^n) = \sigma(\mathcal{I}^n),$$

where $\mathcal{I}^n := \left\{\prod_{j=1}^n I_j \mid I_j\text{'s are subintervals of } [0,1].\right\}$ denotes the set of all rectangles in $[0,1]^n$. It could be proved that there exists a unique probability measure $P : \mathcal{B} \to [0,1]$ such that

$$P\left(\prod_{j=1}^n I_j\right) = (b_j - a_j) \times \cdots \times (b_1 - a_1)$$

for any intervals I_j's having endpoints $0 \le a \le b \le 1$. It is important to note that $P(A)$ coincides with the n-dimensional volume of any event $A \in \mathcal{B}([0,1]^n)$.

Example 4.
Consider the Borel construction $([0,1], \mathcal{B}, P)$.

Calculate the probability that a randomly selected point $\omega \in [0,1]$ belongs to the set $A = [0, 0.1) \cup (0.4, 0.6) \cup (0.9, 1]$.

Example 5.
Consider the Borel construction $([0,1]^2, \mathcal{B}, P)$, and let

$A = [0, 0.5] \times [0.5, 1] \cup [0.5, 1] \times [0, 0.5]$.

Calculate the probability that a randomly selected point $\omega \in [0,1]^2$ belongs to the set A.

Some properties

Complmenting [Prove the proposition.]

Proposition 6.

Let $(\Omega, \mathfrak{F}, P)$ *be a probability space. Then*

(a) $P(A^{\complement}) = 1 - P(A)$ *for any* $A \in \mathfrak{F}$.

(b) $P(\varnothing) = 0$.

Example 7.
The probability of **getting at least one tail** in the trial of 10 coin tosses.

Example 8.
A company has 120 employees that owns a car. Amont them, 80 of them either use a Toyota or a Honda. Find the probability that an employee **does not drive neither Toyota nor Honda**.

Monotonicity [Prove the proposition.]

Proposition 9 (Monotonicity.).
Let $(\Omega, \mathfrak{F}, P)$ be a probability space and $A, B \in \mathfrak{F}$. Then

$$A \subset B \implies P(A) \leq P(B).$$

Example 10.
The probability of drawing a **club** is less than of drawing a **black card**.

Proposition 11 (Sub-σ-additivity.).
Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{A_i\}_{i\in\mathbb{N}}$ be a countable collection of events in $\mathcal{F}$. Then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i).$$

Example 12.
A health dataset collected from 100 patients shows that 40 of them have diabetes and 30 of them have high cholesterol. We may conclude that **no more than** 70 **of them have at least one of the two problems**.

Monotone sequences [Prove the results.]

Proposition 13.
Let $(\Omega, \mathfrak{F}, P)$ *be a probability space and* $\{A_i\}_{i\in\mathbb{N}}$ *be a nondecreasing sequence of events in* $\mathfrak{F}$*, i.e.* $A_i \subset A_{i+1}$ *for all* $i \in \mathbb{N}$*. Then*

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} P(A_i).$$

Corollary 14.
Let $(\Omega, \mathfrak{F}, P)$ *be a probability space and* $\{A_i\}_{i\in\mathbb{N}}$ *be a nonincreasing sequence of events in* $\mathfrak{F}$*, i.e.* $A_{i+1} \subset A_i$ *for all* $i \in \mathbb{N}$*. Then*

$$P\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} P(A_i).$$

Poincaré formula [Prove the proposition.]

The Poincaré formula is considered as the most important ones since it allows to compute the probabilities of the **and**/**or** events.

From now on, let $(\omega, \{, p)$ be a probability space.

Proposition 15 (Poincaré formula for 2 sets).
Let $A, B \in \mathfrak{F}$ be two events. Then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

An example using Poincaré formula [Verify.]

Example 16 (SOA Exam P Sample Questions).
The probability that a visit to a primary care physician's (PCP) office results in neither labwork nor referral to a specialist is 35%. Of those coming to a PCP's office, 30% are referred to specialists and 40% require lab work. Calculate the probability that a visit to a PCP's office results in both lab work and referral to a specialist.

Poincaré formula for 3 sets [Prove the proposition.]

Proposition 17 (Poincaré formula for 3 sets).
Let A_1, A_2, A_3 be three events. Then

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3) = & P(A_1) + P(A_2) + P(A_3) \\ & - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3). \end{aligned}$$

An example using Poincaré formula for 3 sets [Verify.]

Example 18 (SOA Exam P Sample Questions).
A survey of a group's viewing habits over the last year revealed the following information:

(i) 28% watched gynmastics
(ii) 29% watched baseball
(iii) 19% watched soccer
(iv) 14% watched gynmastics and baseball
(v) 12% watched baseball and soccer
(vi) 10% watched gynmastics and soccer
(vii) 8% watched all three sports.

Calculate the percentage of the group that watched none of the three sports during the last year.

Takeaways

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- The generated σ-field has no closed-form description.
- One should become more familiar with the formal definition of a probability space.
- One should be able to use basic properties of a probability measure, with a special emphasis on the Poincaré formula.


Bon journée!