A formal introduction to probability spaces

MTH382 Probability Theory for Finance and Actuarial Science

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Overview

In this class, we shall study a formal definition of a **probability space** using the language of **measure theory**.

It is with this level of precision that allows the whole mathematical theory of probability to be developed.

What is inside ?

The σ-filed of events.

Probability measures

Some properties

The σ-filed of events.

Outcomes and Events

- An **outcome** is a result of a random phenomena or a random experiment. The set of all possible outcomes ω is called the **sample space** Ω.
- An **event** is a subset of Ω that represents a situation in which one could observe and that a **probability** could be assigned. (More precise definition is coming...)
- An outcome ω is said to **realize** an event A if $\omega \in A$.

Definition 1.
Let Ω be a nonempty set. A family $\mathcal{F} \subset 2^{\Omega}$ is called a **σ-field** (also called a **σ-algebra**) over Ω if the following conditions are satisfied:

(σ1) $\Omega \in \mathcal{F}$.
(σ2) If $A \in \mathcal{F}$, then $A^{\complement} := \Omega \setminus A \in \mathcal{F}$.
(σ3) If $A_i \in \mathcal{F}$ for each $i \in \mathbb{N}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is called a **measurable space**, the set Ω is called the **sample space**, and elements of $\mathcal{F}$ are called **events**.

Some immediate properties [Prove the proposition.]

Proposition 2.

Let $(\Omega, \mathfrak{F})$ be a measurable space. Then the following conditions hold:

(a) $\varnothing \in \mathfrak{F}$.

(b) *If $A_i \in \mathfrak{F}$ for each $i \in \mathbb{N}$, then $\bigcap_{i=1}^{\infty} \in \mathfrak{F}$.*

(c) *If $A, B \in \mathfrak{F}$, then $A \cup B \in \mathfrak{F}$ and $A \cap B \in \mathfrak{F}$.*

(d) *If $A, B \in \mathfrak{F}$, then $A \setminus B \in \mathfrak{F}$.*

Some immediate examples [Verify each one.]

Example 3.

Let Ω be a nonempty set.

- $\mathcal{F}_0 = \{\varnothing, \Omega\}$ is the smallest σ-field over Ω, known as the **trivial σ-field**.
- $\mathcal{F}_\infty = 2^\Omega$ is the largest σ-field over Ω, known as the **gross σ-field**.

Here, $\mathcal{F}_0$ and $\mathcal{F}_\infty$ are the **weakest** and the **strongest** σ-fields, respectively, in the sense that

$$\mathcal{F}_0 \subset \mathcal{F} \subset \mathcal{F}_\infty$$

for any σ-field $\mathcal{F}$ over Ω.

- Let $\Omega = \{a, b, c, d\}$. Then $\mathcal{F} = \{\varnothing, \{a, b\}, \{c, d\}, \Omega\}$ is a σ-field over Ω.
- For any Ω and $A \subset \Omega$, the set $\{\varnothing, A, A^\complement, \Omega\}$ is a σ-field.
- For any Ω and a nonempty subset $A \subset \Omega$, the family $\{\varnothing, A, \Omega\}$ is **not** a σ-field.
- Consider an interval $\Omega = (0, 1]$. Then

Proposition 4.

Let Ω be a nonempty set and let

$$\mathcal{A} = \left\{ A \subset \Omega \mid \textit{either } A \textit{ or } A^{\complement} \textit{ is countable} \right\}.$$

Then $\mathcal{A}$ is a σ-field over Ω.

Some more sophisticated examples [Left as exercises.]

Proposition 5 (Trace σ-field).
Let $(\Omega, \mathcal{F})$ be a measurable space and $E \subset \Omega$ is a given nonempty set. Then

$$\mathcal{F}_E := \{E \cap A \mid A \in \mathcal{F}\}$$

is a σ-field over Ω.

Proposition 6 (Pre-image σ-field).
Let $(\Omega', \mathcal{F}')$ be a measurable space, Ω a nonempty set, and $f : \Omega \to \Omega'$ a given map. Then

$$\mathcal{F}_f := \{f^{-1}(A') \mid A' \in \mathcal{F}'\}$$

is a σ-field. Then is a σ-field over Ω

Example 7.
Suppose that $\Omega' = \{0, 1, 2\}$ be equipped with a σ-field $\mathcal{F}' = 2^{\Omega}$.
Let $\Omega = \{(\mathrm{H}, \mathrm{H}), (\mathrm{H}, \mathrm{T}), (\mathrm{T}, \mathrm{H}), (\mathrm{T}, \mathrm{T})\}$ and define $f : \Omega \to \Omega'$ by

$$f((\mathrm{H}, \mathrm{H})) = 2, \quad f((\mathrm{H}, \mathrm{T})) = f((\mathrm{T}, \mathrm{H})) = 1, \quad f((\mathrm{T}, \mathrm{T})) = 0.$$

Calculate the σ-field $\mathcal{F}_f$.

Generated σ-field [Prove the propositions.]

Fix a set Ω. In practice, the σ-field $\mathcal{F}$ over Ω is usually not given, but rather be constructed from some initial information — the family $\mathcal{E} \subset 2^{\Omega}$ of **base events**. We have already seen this earlier in many of our examples in the previous classes.

Proposition 8.

Suppose that Ω is a nonempty set and $\mathcal{E} \subset 2^{\Omega}$ be a nonempty family of subsets of Ω. Then there exists a unique smallest σ-field over Ω containing $\mathcal{E}$, defined by

$$\sigma(\mathcal{E}) := \bigcap \{\mathcal{F} \subset 2^{\Omega} \mid \mathcal{F} \supset \mathcal{E} \text{ is a } \sigma\text{-field over } \Omega.\} \tag{1}$$

The proof of the above proposition is based on the following fact.

Proposition 9.

Let Ω be a nonempty set and $\{\mathcal{F}_\lambda\}_{\lambda \in \Lambda}$ be a collection of σ-fields over Ω. Then $\mathcal{F} := \bigcap_{\lambda \in \Lambda} \mathcal{F}_\lambda$ is also a σ-field over Ω.

Definition 10.
Suppose that Ω is nonempty and let $\mathcal{E} \subset 2^{\Omega}$ be nonempty family containing **base events**. Then the collection $\sigma(\mathcal{E})$, as defined in (1), is called the **σ-field generated by** $\mathcal{E}$. On the other hand, if $\mathcal{F}$ is a σ-field over Ω such that $\mathcal{F} = \sigma(\mathcal{E})$ for some collection of sets $\mathcal{E} \subset 2^{\Omega}$, then $\mathcal{F}$ is said to be **generated by** $\mathcal{E}$.

It is unfortunate that $\sigma(\mathcal{E})$ has no explicit or constructive formula. Therefore it is usually not possible to make a descriptive list of its elements. One should have seen from the earlier examples that even in the case where Ω is finite, the enumeration of $\sigma(\mathcal{E})$ is a tough job. One might think that $\sigma(\mathcal{E})$ could be constructed, for any $\mathcal{E}$, by adding to the family $\mathcal{E}$ all possible countable unions of its members and complements. But this it not true, even when we repeat such a process uncountably.

Some examples [Verify each one.]

Example 11.
Let Ω be a finite set, and $\mathcal{E} = \{\{\omega\} \mid \omega \in \Omega\}$. Then $\sigma(\mathcal{E}) = 2^\Omega$.

Example 12.
Let $\Omega = \{a, b, c, d\}$ and $\mathcal{E} = \{\{a, b\}, \{b, c\}\}$. Then $\sigma(\mathcal{E}) = 2^\Omega$.

More generally, one could come up with the following result.

Proposition 13.

Let Ω be given, and Ω is partitioned into $A_1, A_2, \ldots, A_N$ (which means $\Omega = A_1 \uplus \cdots \uplus A_N$). Then $\#\sigma(\{A_1, \ldots, A_N\}) = 2^N$.

Probability measures

Definition 14.
Let $(\Omega, \mathcal{F})$ be a measurable space. A **probability measure** (or simply a **probability**) on $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \to [0, 1]$ such that the following conditions are satisfied:

(P1) $P(\Omega) = 1$.

(P2) For any sequence of events $(A_i)_{i \in \mathbb{N}}$ in $\mathcal{E}$ such that $A_i \cap A_j = \varnothing$ for all $i, j \in \mathbb{N}$ with $i \neq j$, it holds

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

Whenever P is a probability on $(\Omega, \mathcal{F})$, the tripple $(\Omega, \mathcal{F}, P)$ is called a **probability space**.

Some problems utilizing (P1) and (P2) [Verify each one.]

Example 15 (SOA Exam P Sample Question).
You are given that $P(A \cup B) = 0.7$ and $P(A \cup B^{\complement}) = 0.9$.
Calculate $P(A)$.

Example 16 (SOA Exam P Sample Question).
In modeling the number of claims filed by an individual under an automobile policy during a three-year period, an actuary makes the simplifying assumption that for all integers $n \geq 0$, $p(n+1) = 0.2p(n)$ where $p(n)$ represents the probability that the policyholder files n claims during the period.
Calculate the probability that a policyholder files more than one claim.

Borel's construction (1D case)

In the unit interval $[0, 1]$, we put $\mathcal{I} := \{(a, b) \mid 0 \leq a < b \leq 1\}$ to be the set of all open subintervals of $[0, 1]$.

Consider $\Omega = [0, 1]$. The Borel σ-field, denoted by $\mathcal{B}$ (or $\mathcal{B}([0, 1])$), is defined by

$$\mathcal{B} = \sigma(\mathcal{I}).$$

Then there exists a unique probability measure $P : \mathcal{B} \to [0, 1]$ such that

$$P(I) = b - a, .$$

for any subinterval $I \subset [0, 1]$ with endpoints $a \leq b$.

Borel's construction (nD case)

A similar result can be carried out in the n-dimensional case.

Consider $\Omega = [0,1]^n$. The **Borel σ-field** over $[0,1]^n = \underbrace{[0,1] \times \cdots \times [0,1]}_{n \text{ times}}$, denoted $\mathcal{B}$ (or $\mathcal{B}([0,1]^n)$), is defined by

$$\mathcal{B}([0,1]^n) = \sigma(\mathcal{I}^n),$$

where $\mathcal{I}^n := \left\{ \prod_{j=1}^n I_j \mid I_j\text{'s are subintervals of } [0,1]. \right\}$ denotes the set of all rectangles in $[0,1]^n$. It could be proved that there exists a unique probability measure $P : \mathcal{B} \to [0,1]$ such that

$$P\left(\prod_{j=1}^n I_j\right) = (b_j - a_j) \times \cdots \times (b_1 - a_1)$$

for any intervals I_j's having endpoints $0 \le a \le b \le 1$. It is important to note that $P(A)$ coincides with the n-dimensional volume of any event $A \in \mathcal{B}([0,1]^n)$.

Borel's construction in action

Example 17.
Consider the Borel construction $([0, 1], \mathcal{B}, P)$.

Calculate the probability that a randomly selected point $\omega \in [0, 1]$ belongs to the set $A = [0, 0.1) \cup (0.4, 0.6) \cup (0.9, 1]$.

Example 18.
Consider the Borel construction $([0, 1]^2, \mathcal{B}, P)$, and let

$A = [0, 0.5] \times [0.5, 1] \cup [0.5, 1] \times [0, 0.5]$.

Calculate the probability that a randomly selected point $\omega \in [0, 1]^2$ belongs to the set A.

Some properties

Complmenting [Prove the proposition.]

Proposition 19.
Let $(\Omega, \mathcal{F}, P)$ be a probability space. Then

(a) $P(A^{\complement}) = 1 - P(A)$ *for any* $A \in \mathcal{F}$.
(b) $P(\varnothing) = 0$.

Example 20.

- The probability of **not getting a tail** in the trial of 10 coin tosses.
- The probability of getting **head and tail simultaneously** in a single coin toss.

Monotonicity [Prove the proposition.]

Proposition 21 (Monotonicity.).
Let $(\Omega, \mathfrak{F}, P)$ *be a probability space and* $A, B \in \mathfrak{F}$. *Then*

$$A \subset B \implies P(A) \leq P(B).$$

Example 22.
The probability of drawing a **club** is less than of drawing a **black card**.

Proposition 23 (Sub-σ-additivity.).
Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{A_i\}_{i\in\mathbb{N}}$ be a countable collection of events in $\mathcal{F}$. Then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i).$$

Example 24 (Hedging by combination).
The risk (**i.e.** probability) of **losing money invested in both assets** cannot be worse than the sum of the **individual risks** of the two assets.

Monotone sequences [Prove the results.]

Proposition 25.
Let $(\Omega, \mathcal{F}, P)$ *be a probability space and* $\{A_i\}_{i\in\mathbb{N}}$ *be a nondecreasing sequence of events in* $\mathcal{F}$*, i.e.* $A_i \subset A_{i+1}$ *for all* $i \in \mathbb{N}$*. Then*

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} P(A_i).$$

Corollary 26.
Let $(\Omega, \mathcal{F}, P)$ *be a probability space and* $\{A_i\}_{i\in\mathbb{N}}$ *be a nonincreasing sequence of events in* $\mathcal{F}$*, i.e.* $A_{i+1} \subset A_i$ *for all* $i \in \mathbb{N}$*. Then*

$$P\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} P(A_i).$$

Poincaré formula [Prove the proposition.]

The Poincaré formula is considered as the most important ones since it allows to compute the probabilities of the **and**/**or** events.

From now on, let $(\omega, \{, p)$ be a probability space.

Proposition 27 (Poincaré formula for 2 **sets).**
Let $A, B \in \mathfrak{F}$ *be two events. Then* $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

An example using Poincaré formula [Verify.]

Example 28 (SOA Exam P Sample Questions).
The probability that a visit to a primary care physician's (PCP) office results in neither labwork nor referral to a specialist is 35%. Of those coming to a PCP's office, 30% are referred to specialists and 40% require lab work. Calculate the probability that a visit to a PCP's office results in both lab work and referral to a specialist.

Poincaré formula for 3 sets [Prove the proposition.]

Proposition 29 (Poincaré formula for 3 sets).
Let A_1, A_2, A_3 be three events. Then

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3) = & P(A_1) + P(A_2) + P(A_3) \\ & - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3). \end{aligned}$$

An example using Poincaré formula for 3 sets [Verify.]

Example 30 (SOA Exam P Sample Questions).
A survey of a group's viewing habits over the last year revealed the following information:

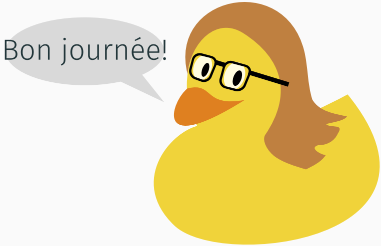
(i) 28% watched gymnastics
(ii) 29% watched baseball
(iii) 19% watched soccer
(iv) 14% watched gymnastics and baseball
(v) 12% watched baseball and soccer
(vi) 10% watched gymnastics and soccer
(vii) 8% watched all three sports.

Calculate the percentage of the group that watched none of the three sports during the last year.

Takeaways

Takeaways

- The generated σ-field has no closed-form description.
- One should become more familiar with the formal definition of a probability space.
- One should be able to use basic properties of a probability measure, with a special emphasis on the Poincaré formula.


Bon journée!