Continuous random variables — Some common distributions

MTH382 Probability Theory for Finance and Actuarial Science

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Overview

In this lecture, we make an explore some common distributions of continuous random variables. The most notable ones are the uniform and normal (or Gaussian) distributions.

Uniform distributions

The idea of the uniform distribution in the continuous setting is somehow similar to the discrete one. In this continuous setting, uniform distribution means that all points in a designated interval is distributed evenly.

Definition 1.
A random variable X is said to have the **uniform distribution** over the interval $[a, b]$ if its density function is given by

$$f(x) = \begin{cases} \frac{1}{b-1} & \text{if } a < x < b, \\ 0 & \text{otherwise.} \end{cases}$$

We write $X \sim \mathrm{Uniform}(a, b)$.

Example

A typical example of a r.v. having a uniform distribution was already given in previous slides (which one ?).

Expectation and variance

Theorem 2.
If $X \sim \mathrm{Uniform}(a, b)$*, then*

$$\mathbb{E}X = \frac{a+b}{2}, \quad \textit{and} \quad \mathsf{Var}[X] = \frac{(b-a)^2}{12}.$$

Normal distributions

The normal (or Gaussian) distribution is the one that has been central in several probabilistic and statistical modeling. We begin with the *standard* normal distribution.

Definition 3.
A random variable X is said to have the **standard normal distribution** (or **standard Gaussian distribution**) if its density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

In this case, we also write $X \sim \mathrm{Normal}(0, 1)$.

Expectation and variance

The next theorem explains the numbers 0 and 1 in the notion $\mathrm{Normal}(0, 1)$.

Theorem 4.
If $X \sim \mathrm{Normal}(0, 1)$*, then*

$$\mathbb{E}X = 0, \quad \textit{and} \quad \mathsf{Var}[X] = 1.$$

Normally, there is no reason that a r.v. would needfully has zero expectation and unit variance. This leads to the general concept of the normal distributions.

Definition 5.
If $Z \sim \mathrm{Normal}(0, 1)$ and X is a r.v. in which

$$X = \sigma Z + \mu,$$

for some fixed $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}$. Then X is said to have a **normal distribution** (or **Gaussian distribution**) with parameters μ and σ^2, or symbolically written with $X \sim \mathrm{Normal}(\mu, \sigma^2)$.

Theorem 6.
If $X \sim \mathrm{Normal}(\mu, \sigma^2)$, *then*

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

and

$$\mathbb{E}X = \mu, \quad \textit{and} \quad \mathsf{Var}[X] = \sigma^2.$$

Standardization

Every normally distributed r.v.s could be stardized into a standard normally distributed ones.

Theorem 7.
If $X \sim \mathrm{Normal}(\mu, \sigma^2)$*, then* $Z := \frac{X-\mu}{\sigma} \sim \mathrm{Normal}(0, 1)$.

Calculations for standard normal distribution

Probability calculations for standard normal distribution usually based on its CDF function, especially denoted with Φ, with

$$\Phi(z) := P(Z \leq z).$$

Since there is no closed form formula for Φ, one refers to the **standard normal table**, which gives the pre-calculated values of Φ at different z.

It is useful to note that

$$P(Z \geq z) = 1 - P(Z < z) = 1 - \Phi(z).$$

Example

Example 8.
If Z follows the standard normal distribution, find $P(-1 < Z < 1.25)$.

Example 9.
If X follows the normal distribution with parameters $\mu = 1$ and $\sigma = 1$.

Other distributions

Other distributions

There are also other distributions that might be less used or less known. We collect them here, just to provide as examples.

Let X be an absolutetly continuous r.v., then

- it has the **exponential distribution** with parameter $\lambda > 0$ if its density function is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0, \\ 0 & \text{otherwise,} \end{cases}$$

- it has the **gamma distribution** with parameters $\alpha, \lambda > 0$ if its density function is given by

$$f(x) = \begin{cases} \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} & \text{if } x > 0, \\ 0 & \text{otherwise,} \end{cases}$$

  where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$.

Takeaways

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- One should gets familiar with the uniform and normal distributions.
- Any normal distribution with non-standard parameters ($\mu \neq 0$ or $\sigma^2 \neq 1$) could be stadardized.


Bon journée!