Continuous random variables — Definitions and properties

MTH382 Probability Theory for Finance and Actuarial Science

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Overview

This lecture in an introduction to continuous random variables, with a focus on those with density functions. One would see that, while ther are differences to the discrete ones, many of the properties are still valid in this continuous settings.

Definitions

Continuous random variables

Always consider a measurable space $(\Omega, \mathcal{F})$.

Recall that a *discrete* r.v. are the functions from Ω into a countable set E. Most of the time, it is useful to have $E \subset \mathbb{R}$. On the contrary, a *continuous* r.v. takes values on an uncountable set in $\mathbb{R}$. We also require that every *cumulative sets* are actually *events*.

Rather than defining a **continuous** random variable, we give a **general** definition of a random variable that takes possibly **uncountable** values.

Definition 1.
A function $X : \Omega \to \mathbb{R}$ is called a **random variable** (or briefly, a **r.v.**) if

$$\{X \leq x\} := \{\omega \in \Omega \mid X(\omega) \leq x\} \in \mathcal{F}$$

for every $x \in \mathbb{R}$.

Remark.
The definition above is the minimal one that at least allow assigning a probability to each set of the form $\{X \leq x\}$. One should notice that a discrete r.v.s defined in the earlier lectures also satisfy the above definition.

Example 2.
Take $\Omega = [0, 1]$ and $\mathcal{F} = \mathcal{B}([0, 1])$ (recall that $\mathcal{B}([0, 1])$ includes all kinds of intervals). Define $X : \Omega \to \mathbb{R}$ by

$$X(\omega) = \omega^2.$$

Show that X is a r.v. which is not discrete.

Cumulative distribution functions

From now, suppose that P is a probability on $(\Omega, \mathcal{F})$.

The cumulative distribution function of a random variable plays a very important role when one goes beyond discrete r.v.s.

Definition 3.
The **cumulative distribution function** (briefly, **CDF**) of a r.v. X is $F_X : \mathbb{R} \to \mathbb{R}$ given by

$$F_X(x) := P(X \leq x).$$

Example 4.
Find the CDF of the following discrete r.v.s that are represented by their distributions functions:

- $\pi_X(x) = p$ if $x = 1$, $\pi_X(x) = 1 - p$ if $x = 0$, and $\pi_X(x) = 0$ otherwise.
- $\pi_X(x) = 1/10$ for $x = 0, 1, 2, \ldots, 9$ and $\pi_X(x) = 0$ otherwise.

Theorem 5.
A CDF F_X of a r.v. $X : \Omega \to \mathbb{R}$ has has the following properties:

(a) $F : \mathbb{R} \to [0, 1]$,

(b) *F is non-decreasing, i.e.* $x \leq y \implies F(x) \leq Fy$,

(c) *F is right-continuous, i.e.* $\lim_{x \to a^+} F(x) = F(a)$ *for all* $a \in \mathbb{R}$,

(d) *for any* $a \in \mathbb{R}$, $P(X = a) = F(a) - F(a-)$ *where* $F(x-) := \lim_{x \to a^-} F(x)$.

(e) *F has at most countably many points of discontinuity* $d_1, d_2, \ldots,$ *and it could be decomposed into*

$$F(x) = F_d(x) + F_c(x),$$

where F_d *denotes the* **discontinuous** *part and* F_c *is the* **continuous** *part of F.*

(Absolutely) continuous random variables

Definition 6.
A r.v. $X : \Omega \to \mathbb{R}$ is said to be

- **continuous** if its CDF F_X is continuous,
- **absolutely continuous** if there exists an integrable function $f : \mathbb{R} \to \mathbb{R}$, called the **density function** of X, in which

$$F_X(X \leq a) = \int_{-\infty}^{a} f(x)dx.$$

Theorem 7.
Every absolutely continuous r.v. is continuous.

Remark.
There are continuous r.v.s that are not absolutely continuous. However we focus mainly on the absolutely continuous ones (those with densities) here.

Example 8.
Construct the CDF from the following density functions and observe the continuity.

- $f(x) = 1$ if $x \in [-1, -1/2] \cup [1/2, 1]$ and $f(x) = 0$ otherwise.
- $f(x) = \max\{1 - x, 0\}$ for $x \in \mathbb{R}$.

Zero-probability events

A continuous r.v. could be characterized by its values being zero probable.

Theorem 9.
A r.v. $X : \Omega \to \mathbb{R}$ *is continuous if and only if* $P(X = x) = 0$ *for all* $x \in \mathbb{R}$.

This theorem says that a continuous r.v. never has a probability at any of its values on their own, which is a bit counter-intuitive.

The following is a typical example that illustrates the difference between an impossible event and a zero-probability event.

Example 10.
Consider randomly tossing a pin onto a ruler. If X is the r.v. representing the place where the pin drops on the ruler, then X is absolutely continuous and the probability that a pin drops at any point on the ruler equals to 0.

Concepts from previous expeditions

General concepts

It is important that the concepts defined earlier for discrete r.v.s remain valid *with a small tweak* in the setting of a general r.v.s.

We give an example here of how the concept of independence is extended to general r.v.s.

Definition 11 (Independence for general r.v.s.).
Let $X_1, X_2 : \Omega \to \mathbb{R}$ be two r.v.s, then they are **independent** if $P(X_1 \leq x_1, X_2 \leq x_2) = P(X_1 \leq x_1)P(X_2 \leq x_2)$ for any $x_1, x_2 \in \mathbb{R}$.

Example 12.
Consider the pin dropping from Example 10, but with two drops. Let X_1 and X_2 be two absolutely continuous r.v.s representing the position of the first and second drops, respectively. Then X_1 and X_2 are independent.

The most important concepts regarding a r.v. are still the expectation and variance. For absolutely continuous r.v.s, we define them using integrals.

Definition 13.
Let X be an absolutely continuous r.v. with density f. The **expectation** (or **expected value**) of X is defined by

$$\mathbb{E}X := \int_{-\infty}^{\infty} xf(x)dx,$$

whenever the above integral is well-defined.

The same properties could be proved for absolutely continuous r.v.s.

Theorem 14.
Let X_1 and X_2 be two absolutely continuous r.v.s. Then

- $\mathbb{E}[\lambda_1 X_1 + \lambda_2 X_2] = \lambda_1 \mathbb{E}X_1 + \lambda_2 \mathbb{E}X_2$ *for any* $\lambda_1, \lambda_2 \in \mathbb{R}$,
- *if* $X_1 \leq X_2$, *then* $\mathbb{E}X_1 \leq \mathbb{E}X_2$,
- $|\mathbb{E}X| \leq \mathbb{E}[|X|]$.

The variance is given by the same formular as the discrete case. Moreover, the simplified calculation is still applied.

Definition 15.
Suppose that X is an absolutely continuous r.v. with expectation $\mu = \mathbb{E}X$, then its **variance** is

$$\mathrm{Var}[X] := \mathbb{E}[(X-\mu)^2] = \mathbb{E}[X^2] - \mu^2.$$

Example 16.
Let X be a continuous r.v. whose density function is given by

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } 0 \leq x < 1 \\ \frac{2}{3} - \frac{x}{6} & \text{if } 1 \leq x < 4 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the formula for the CDF of X.

(b) Calculate $P(0.5 \leq X \leq 2)$.

(c) Calculate $\mathbb{E}X$ and $\mathrm{Var}[X]$.

Takeaways

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- A general r.v. has the general properties mostly like the discrete ones.
- Most concepts studied earlier about a discrete r.v. extend to the general case.
- Zero-probability events may occur.
- Absolutely continuous r.v.s are captured with its density function.


Bon journée!