Optimization

Lecture 11: Linear programming — Duality

Parin Chaipunya
KMUTT
└ Mathematics @ Faculty of Science
└ The Joint Graduate School of Energy and Environments

Areas of research:

- Multi-agent optimization: Bilevel programs, Game theory
- Optimization modeling: mainly focused on energy and environmental applications

parin.cha@kmutt.ac.th
parinchaipunya.com
github.com/parinchaipunya

Last update: Januaray 2026

Table of contents

Section 1

LP duality

Subsection 1

Dual problems

Dual pair

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $f \in \mathbb{R}^n$. Then the following pair of two LPs is called a **dual pair**

$$
\begin{cases}
\min\limits_{x} & f^t x \qquad (1a) \\
\text{s.t.} & Ax \geq b \qquad (1b) \\
 & x \geq 0 \qquad (1c)
\end{cases}
\qquad
\begin{cases}
\max\limits_{y} & b^t y \qquad (2a) \\
\text{s.t.} & A^t y \leq f \qquad (2b) \\
 & y \geq 0 \qquad (2c)
\end{cases}
$$

In general, one of the two problems will be referred to as the **primal problem** and the other one as the **dual problem**. That is, the **problem** (2) **is dual to** (1) and at the same time the **problem** (1) **is dual to** (2). However, there is no common preference that one of them is identified as the primal problem over the other one.

Since we frame our theory primarily for minimization, we shall take (1) as the primal problem and (2) as the dual problem in these slides.

Note that the primal problem (1) has n variables and m constraints, while the dual problem (2) has m variables and n constraints.

Computational advantage

Recall that the simplex tableau of the primal problem (1) is a $(m+1) \times (m+n+1)$ matrix, while the one of the dual problem (2) is of dimension $(n+1) \times (m+n+1)$.

If a problem has number of variables much smaller than the number of constraints, then the dual problem has a computational advantage over the primal one.

Howoever, to be convinced that the dual problem could be considered instead of its primal one, we have to look at the duality relations.

Subsection 2

Duality relations

Duality relations

There are a number of relationships between the two problems in duality. Let us write the optimal values of (1) and (2)

$$p^* = \inf\{f^t x \mid Ax \geq b,\ x \geq 0\} \quad \text{and} \quad d^* = \sup\{b^t y \mid A^t y \leq f,\ y \geq 0\}\,.$$

Note that the two optimal values could be infinite.

Duality relations

Theorem 1

The following statements are true.

(a) *If one problem is feasible but has no finite optimal value, then the other problem is infeasible.*

(b) *If both problems are feasible, then both problems have optimal solutions.*

(c) ***Weak duality.*** *If x and y are feasible points of* (1) *and* (2)*, respectively, then*

$$f^t x \geq b^t y \,.$$

(d) ***Strong duality.*** *If one of the problems has an optimal solution, so does the other one. Moreover the two optimal values are equal, that is*

$$p^* = d^* \,.$$

Duality relations

Proof of Theorem 1. (Part 1/3)
(c) From the assumptions,

$$\begin{aligned} f \geq A^t y \text{ and } x \geq 0 \quad &\Longrightarrow \quad f^t x \geq y^t A x \\ Ax \geq b \text{ and } y \geq 0 \quad &\Longrightarrow \quad x^t A^t y \geq b^t y\,. \end{aligned}$$

Since $y^t A x = x^t A^t y$, we obtain

$$f^t x \geq b^t y\,.$$

(a) Assume (1) is feasible with $p^* = -\infty$ but (2) is feasible. Then the weak duality says

$$-\infty \geq b^t y,$$

which is a contradiction. Hence (2) cannot be feasible.

Proof of Theorem 1. (Part 2/3)
(b) From the weak duality, $p^* > -\infty$. Assume that (1) has no optimal solution, which means that there is no feasible point x in which $f^t x = p^*$. Define $T^* = \{x \mid f^t x = p^*\}$ and note that it is a nonempty convex set disjoint from the nonempty convex feasible set $C = \{x \mid Ax \geq b,\ x \geq 0\}$. The Hahn-Banach separation theorem* says that there is a hyperplane S separating T^* and C. Since T^* is itself a hyperplane, S must be parallel to T^* and hence we could see that $S = \{x \mid f^t x = p^* + \varepsilon\}$ for some $\varepsilon > 0$. This means

$$p^* + \varepsilon < f^t x \quad \forall x \in C\,,$$

so that p^* cannot be the infimum over C — a contradiction. Thus we conclude that (1) has an optimal solution.

***The Hahn-Banach separation theorem.** If A and B are closed convex subsets of $\mathbb{R}^n$, then there exists a nonzero vector v and $\eta \in \mathbb{R}$ such that $v^t x < \eta < v^t y$ for all $x \in A$ and $y \in B$.

Duality relations

Proof of Theorem 1. (Part 3/3)

(d) Let x^* be an optimal solution of (1). Then the KKT conditions implies the existence of $\lambda^* \in \mathbb{R}_+^m$ and $\mu^* \in \mathbb{R}_+^n$ such that

$$f - A^t\lambda^* - \mu^* = 0 \tag{3}$$

$$(b_j - [Ax^*]_{j,.})\lambda_j^* = 0 \quad (\forall j = 1, \dots, m) \tag{4}$$

$$\mu_i^* x_i^* = 0 \quad (\forall i = 1, \dots, n)\,. \tag{5}$$

The equation (3) implies that $f - A^t\lambda^* = \mu^* \geq 0$, and hence λ^* is feasible for the problem (2). Combining (3) and (5), one obtains

$$f^t x^* - (\lambda^*)^t A x^* = f^t x^* - (\lambda^*)^t A x^* - (\mu^*)^t x^* = 0\,,$$

which gives $f^t x^* = (\lambda^*)^t A x^*$. Together with the equation (4), it implies that

$$0 = (b - Ax^*)^t \lambda^* = b^t\lambda^* - (\lambda^*)^t A x^* = b^t \lambda^* - f^t x^*\,.$$

Using weak duality, one obtains

$$b^t\lambda^* = f^t x^* \geq b^t y$$

for any choice of y that is feasible for the problem (2). Finally, we conclude that (2) has an optimal solution $y^* = \lambda^*$. □

Duality relations

One should have already noticed, at this stage, that the **dual optimal solution** y^* **of the problem** (2) **is actually the Lagrange multipliers** λ^* assigned to the primal linear inequality constraints (1b).

On the other hand, the **primal optimal solution** x^* **also acts as the Lagrange multipliers for the constraints** (2b) in the same way as described above.

Duality relations

The duality relations established in Theorem 1 provide a rigorous justification for investigating the dual problems.

Situations that we may transition to the dual approach include simplifying algorithmic implementation, obtaining certificates of optimality via weak duality, and conducting rigorous sensitivity analysis on the constraint parameters.

Section 2

Duality of the simplex tableaux

An observation

Consider the following dual pair.

$$\begin{cases} \min_{x} & 6x_1 + 2x_2 + 3x_3 & & (6a) \\ \text{s.t.} & x_1 + x_2 & \geq 2 & (6b) \\ & x_1 - x_2 + x_3 & \geq 1 & (6c) \\ & x_1, x_2 & \geq 0 & (6d) \end{cases}$$

$$\begin{cases} \max_{y} & 2y_1 + y_2 & & (7a) \\ \text{s.t.} & y_1 + y_2 & \leq 6 & (7b) \\ & y_1 - y_2 & \leq 2 & (7c) \\ & x_2 & \leq 3 & (7d) \\ & y_1, y_2 & \geq 0 & (7e) \end{cases}$$

An observation

Let the following M and T correspond to the problems (6) and (7), respectively.

$M =$

x_1	x_2	x_3	s_1	s_2	
6	2	3	0	0	0
−1	−1	0	1	0	−2
−1	1	−1	0	1	−1

$T =$

y_1	y_2	w_1	w_2	w_3	
−2	−1	0	0	0	0
1	1	1	0	0	6
1	−1	0	1	0	2
0	1	0	0	1	3

An observation

After performing the simplex algorithm on both tableaux, we arrive at the following optimal forms.

$M^* =$

x_1	x_2	x_3	s_1	s_2	
0	0	1	4	2	-10
1	0	$1/2$	$-1/2$	$-1/2$	$3/2$
0	1	$-1/2$	$-1/2$	$1/2$	$1/2$

$$\implies (x^*; s^*) = \left(\frac{3}{2}, \frac{1}{2}, 0\ ; 0, 0\right)$$

$T^* =$

y_1	y_2	w_1	w_2	w_3	
0	0	$3/2$	$1/2$	0	10
0	1	$1/2$	$-1/2$	0	2
1	0	$1/2$	$1/2$	0	4
0	0	$-1/2$	$1/2$	1	1

$$\implies (y^*; w^*) = (4, 2\ ; 0, 0, 1)$$

An observation

One may observe that the optimal dual solution y^* could be obtained directly from the optimal simplex tableau M^*. The same holds for x^* and the dual simplex tableau T^*.

More precisely, the *objective coefficients* presented in the *slack sections* of the simplex tableaux at their optimal stages provide information about their optimal dual counterparts.

This is not just a coincidence, but a predictable behavior of the simplex algorithm.

Tableau analysis

Theorem 2

The vector of cost coefficients of the slack variables in an optimal simplex tableau is always an optimal dual solution.

Proof of Theorem 2. (Part 1/3)

Suppose that M is the simplex tableau of (1) written as a matrix which yields an optimal tableau M^*.

Since M^* is obtained from M through a sequence of pivots, it could be represented as

$$M^* = QM\,,$$

where $Q \in \mathbb{R}^{(m+1)\times(m+1)}$ is the of composition of all the performed pivots.

Tableau analysis

Proof of Theorem 1. (Part 2/3)
Since the pivoting element never belongs to the first row (*i.e.* the objective row), we may see that the first column of all the pivot matrices is $[1\ 0\ \ldots 0]^t$. Thus we could write Q as

$$Q = \left[\begin{array}{c|c} 1 & v^t \\ \hline 0 & B \end{array}\right],$$

for some vector $v \in \mathbb{R}^{n+m}$ and $B \in \mathbb{R}^{(n+m)\times(n+m)}$.
Recall also that

$$M = \left[\begin{array}{c|c|c} f^t & 0^t & 0 \\ \hline -A & I & -b \end{array}\right].$$

Tableau analysis

Proof of Theorem 1. (Part 3/3)
Hence, we may write M^* as

$$M = \left[\begin{array}{c|c|c} f^t - v^t A & v^t & -v^t b \\ \hline -BA & B & -Bb \end{array}\right].$$

Since M^* is in an optimal form, we must have

$$v^t \geq 0 \quad \text{and} \quad f^t - v^t A \geq 0\,.$$

This implies that v is feasible for (2). The tablueau M^* also says that $p^* = v^t b$. Knowing also that $p^* = f^t x^*$, the strong duality concludes that v is optimal for the problem (2).

Section 3

Shadow price

Resource utilization

To clearly explain the idea of **shadow prices**, let us take an example from production.

Your firm has three resources R_1, R_2 and R_3 and makes four products P_1, P_2, P_2 and P_4. The following technology table gives a summary of the unitary resource requirements and, resource availability, and selling prices.

	P_1	P_2	P_3	P_4	Available (tons)
R_1	2	1	1	1	30
R_2	1	0	2	1	15
R_3	1	1	1	0	5
Revenue (k$)	6	1	4	5	

Resource utilization

The goal of this problem is to utilize the limited resources the best we can by making the maximum revenue. This is a LP given as follows.

$$\begin{cases} \min & 6x_1 + x_2 + 4x_3 + 5x_4 \\ \text{s.t.} & 2x_1 + x_2 + x_3 + x_4 \leq 30 \\ & x_1 + 2x_3 + x_4 \leq 15 \\ & x_1 + x_2 + x_3 \leq 5 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{cases}$$

Solving this with the simplex method gives, respective, the primal and dual optimal points

$$x^* = (5, 0, 0, 10), \quad \text{and} \quad y^* = (0, 5, 1)\,.$$

Moreover, the maximum revue generated by x^* is 80 k$.

Value of a resource

Now imagine that, prior to your production, someone offered to buy one unit of resource R_2.

The question is: *How much should you sell this unit of resource R_2?*

Since the resource R_2 is used up in your optimal production plan, relinquishing it at a low price would just be detrimental. **Its unitary price should therefore be set no lower than the potential fortune it generates.** This potential is known as the **shadow price**.

In fact, the shadow price of the resource R_2 equals to the dual optimal solution $y_2^* = 5$.

Similarly, the shadow prices of R_3 is $y_3^* = 1$ and of R_1 is $y_1^* = 0$. Note that the reason that the resource R_1 has no value, $x_1^* = 0$, is because it is not used up in the optimal production plan.

Value of a resource

One could also make a formal analysis on the shadow prices from the optimal simplex tableau.

$$M^* = \begin{array}{|ccccccc|c|} \multicolumn{1}{c}{x_1} & x_2 & x_3 & x_4 & s_1 & s_2 & \multicolumn{1}{c}{s_3} & \multicolumn{1}{c}{} \\ \hline 0 & 0 & 7 & 0 & 0 & 5 & 1 & 80 \\ \hline 0 & 0 & -2 & 0 & 1 & -1 & -1 & 10 \\ 0 & -1 & 1 & 1 & 0 & 1 & -1 & 10 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 5 \\ \hline \end{array}$$

Value of a resource

Rearranging s_2 to the RHS, we obtain

$$M' = \begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & s_1 & s_3 & \\ \hline 0 & 0 & 7 & 0 & 0 & 1 & 80 - 5s_2 \\ \hline 0 & 0 & -2 & 0 & 1 & -1 & 10 + s_2 \\ 0 & -1 & 1 & 1 & 0 & -1 & 10 - s_2 \\ 1 & 1 & 1 & 0 & 0 & 1 & 5 \end{array}$$

With any given $s_2 \geq 0$, as long as $10 - s_2 \geq 0$, the tableau remains optimal but the maximum revenue reduces to $80 - 5s_2$.

Value of a resource

Then we should find out the smallest selling price ξ^* of the resource R_2 by considering that the new revenue (production + sales of resrouce R_2) should not go below the original benefit 80 k$:

$$\underbrace{80 - 5s_2}_{\text{production revenue}} + \underbrace{\xi^* \times s_2}_{\text{sales of resource}} \geq \underbrace{80}_{\text{original revenue}},$$

which gives

$$\xi^* \geq 5 = y_2^* .$$

In general, knowing the shadow prices of all the resources helps the decision maker understand which resource is more valuable. This, in turn, says that **losing a unit of resource with higher shadow price is more costly** than the one with lower shadow prices.

That's it!

Key takeaways.

- The dual pair of LPs.
- The relationships between the optimal solutions of a dual pair.
- The relationship between the cost coefficients and Lagrange multipliers.
- Shadow prices.

Thank you.