Optimization

Lecture 10: Linear programming — Simplex methods

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Areas of research:

- Multi-agent optimization: Bilevel programs, Game theory
- Optimization modeling: mainly focused on energy and environmental applications

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Section 1

Linear program revisited

Linear programs

A **linear program** (**LP**) is a problem of the form

$$
\begin{cases}
\min & f_1x_1 + \cdots + f_nx_n + f_0 \\
\text{s.t.} & a_{11}x_1 + \cdots + a_{1n}x_n \quad \leq b_1 \\
& \qquad\qquad\qquad \vdots \\
& a_{r1}x_1 + \cdots + a_{rn}x_n \quad \leq b_r \\
& c_{11}x_1 + \cdots + c_{1n}x_n \quad = d_1 \\
& \qquad\qquad\qquad \vdots \\
& c_{\ell 1}x_1 + \cdots + c_{\ell n}x_n \quad = d_\ell \,.
\end{cases}
$$

Linear Programs

If we write

$$
\begin{aligned}
f &= (f_1, \ldots, f_n)\,, \\
A &= [a_{ji}]\,, \quad b = (b_1, \ldots, b_r)\,, \\
C &= [c_{ki}]\,, \quad d = (d_1, \ldots, d_\ell)\,,
\end{aligned}
$$

then we have a compact representation of a linear program

$$
\begin{cases}
\min & f^t x + f_0 \quad (1) \\
\text{s.t.} & Ax \leq b \quad (2) \\
 & Cx = d \quad (3)
\end{cases}
$$

Existence

Theorem 1

If the LP (1) *is feasible and the objective values are bounded below, i.e.*

$$p^* = \inf\{f^t x \mid Ax \leq b,\ Cx = d\} > -\infty\,,$$

then this LP has an optimal solution.

This theorem then says that an LP either

$$\underbrace{\text{is infeasible}}_{p^*=+\infty}, \quad \text{or} \quad \underbrace{\text{is not bounded below}}_{p^*=-\infty}, \quad \text{or} \quad \underbrace{\text{has an optimal solution.}}_{-\infty<p^*<+\infty}$$

Canonical form

An LP Is said to be in a **canonical form** if it is written as

$$\begin{cases} \min & f^t x + f_0 \\ \text{s.t.} & Cx = d \\ & x \ \geq 0\,, \end{cases}$$

that is, the constraints are all equalities and the variables are all non-negative.

Canonical form

The following is an important fact that allows us to only consider only problems presented in their canonical forms.

Theorem 2

Every LP can be transformed into a canonical form.

Transformation techniques

In particular, a linear inequality

$$a_1x_1 + \cdots + a_nx_n \leq b$$

could be equivalently transformed into

$$a_1x_1 + \cdots + a_nx_n + s = b, \quad s \geq 0\,.$$

Transformation techniques

Lastly, if a variable x is allowed to take negative values, then we introduce two additional variables x^+ and x^- with the conditions

$$x = x^+ - x^-, \quad x^+, x^- \geq 0\,.$$

One may need to combine this variable splitting with the previous technique, if necessary, to obtain an equivalent reformulations.

An example

Example 3

The following LP on the left could be converted into a canonical form on the right:

$$
\begin{cases}
\min & 2x_1 + x_2 - 2x_3 \\
\text{s.t.} & x_1 + x_2 = 5 \\
& 3x_1 + 2x_3 \le 4 \\
& x_1 \ge 0 \\
& x_2 \ge 1 \\
& x_3 \le 2
\end{cases}
\qquad
\begin{cases}
\min & 2x_1 + x_2 - 2x_3^+ + 2x_3^- \\
\text{s.t.} & x_1 + x_2 = 5 \\
& 3x_1 + 2x_3 + s_1 = 4 \\
& x_2 - s_2 = 1 \\
& x_3^+ - x_3^- + s_3 = 2 \\
& x_1, x_2, x_3^+, x_3^-, s_1, s_2, s_3 \ge 0
\end{cases}
$$

Section 2

Basic solutions of a linear system

LP in a canonical form

Notice that a solution of an LP in a canonical form

$$\begin{cases} \min & f^t x + f_0 & (4a) \\ \text{s.t.} & Cx = d & (4b) \\ & x \geq 0 & (4c) \end{cases}$$

is also a solution of a linear system

$$Cx = d, \quad x \geq 0\,.$$

Suppose that C has a dimension of $m \times n$ and we likely have $m < n$. We shall also make an assumption that C is of a full rank, which means $\text{rank}(A) = m$.
This ensures that $Cx = d$ has a solution, and in fact it always has a **basic solution**.

Basic solutions

Definition 4

Consider a $m \times n$ linear system $Cx = d$ with a full rank matrix C.
Suppose that B is a nonsingular $m \times m$ matrix (this means $\text{rank}(B) = m$) extracted from the columns in $\mathcal{J}_B = (j_1, \ldots, j_m)$ of C, that is

$$B = \begin{bmatrix} | & & | \\ c_{\cdot, j_1} & \cdots & c_{\cdot, j_m} \\ | & & | \end{bmatrix} .$$

Let $x_i^* = 0$ for all $i \notin \mathcal{J}_B$, and let $x_{\mathcal{J}_B}^* = (x_{j_1}^*, \ldots, x_{j_m}^*)$ be the solution of the linear system

$$Bx_{\mathcal{J}_B} = d .$$

Then the vector $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ is called the **basic solution** of $Cx = d$.
Moreover, the variables $x_{j_1}, \ldots, x_{j_m}$ are called the **basic variables** while those x_i with $i \notin \mathcal{J}_B$ are called the **nonbasic variables**.

If a basic solution has some zero components at the basic variables, it is said to be **degenerate**. Otherwise, it is called **non-degenerate**.

Basic solutions

Theorem 5

One could transform, using elemetrary row operations, so that the resulting basic columns becomes an identity matrix.

An example

Consider the system

$$2x_1 + x_2 + 2x_4 = 5$$
$$2x_1 + 2x_2 + x_3 = 5$$
$$2x_1 + x_2 + x_3 + 2x_4 = 6\,.$$

Here, we put

$$C = \begin{bmatrix} 2 & 1 & 0 & 2 \\ 2 & 2 & 1 & 0 \\ 2 & 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad d = \begin{bmatrix} 5 \\ 5 \\ 6 \end{bmatrix}\,.$$

If we pick the first three columns to construct B, that is $\mathcal{J}_B = \{1, 2, 3\}$, we get the matrix

$$B = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}\,.$$

Recall that we need to make sure that $\text{rank}(B) = 3$.

Solving $Bx = d$, we obtain the solution $x^*_{\mathcal{J}_B} = (3, -1, 1)$. Adjoining a zero to the nonbasic variable x_4, we obtain a basic solution $x^* = (3, -1, 1, 0)$ for $Cx = d$.
Note that this is a non-degenerate solution.

An example (Cont.)

Note that we may reduce, using elemetary row operations to obtain the columns of B to an identity matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

An example (Cont.)

Now let's take another B by choosing $j_1 = 2$, $j_2 = 1$ and $j_3 = 3$. Then we have

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Again, this is a viable choice because B has a full rank.

With some steps through elementary row operations on C, we obtain

$$\left[\begin{array}{cccc} 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

If we extract columns $j_1 = 2$, $j_2 = 1$ and $j_3 = 3$, we see that B has been reduced to an identity matrix.

Basic feasible points

In the canonical LP (4), there is no guarantee that a basic solution of $Cx = d$ is feasible as some of the components could be negative.
If a basic solution of $Cx = d$ is also non-negative, it is called a **basic feasible point**.

The following result is central in the development of the simplex algorithms.

Theorem 6 (Fundamental threom of linear programming)

Given any LP in a canonical form (4) where C is a $m \times n$ matrix with rank m. Then, a vector x is an extreme point of the feasible set if and only if it is a basic feasible point.
Moreover, the following statements are true.

- *If there is a feasible point, there is a basic feasible point.*
- *If there is an optimal solution, there is an optimal basic solution.*

Section 3

Simplex algorithm

Simplex tableau

It is traditional to consider the representation of the system

$$Cx = d, \quad x \geq 0$$

as the following tableau

$$\begin{array}{cccc|c}
x_1 & x_2 & \cdots & x_n & \\
\hline
c_{11} & c_{12} & \cdots & c_{1n} & d_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{mn} & d_1 \\
\hline
\end{array}$$

Standing assumptions

Standing assumptions. Always assume that

- C has a dimension of $m \times n$, with $m < n$, and $\text{rank}(C) = m$, and
- the basic feasible points are always non-degenerate.

Note that these assumptions are here just to avoid complicate cases. In practice, a simplex algorithm could be modified so that the standing assumptions are not necessary.

Pivoting

From the tableau, we are said to do a **pivot** around a nonzero entry c_{ki} if the tableau is reduced, using elementary row operations, so that the the entry $\tilde{c}_{ki}$ is 1 and the other entries in column i becomes 0.

$$
\begin{array}{cccccc|c}
x_1 & x_2 & \cdots & x_i & \cdots & x_n & \\
\hline
c_{11} & c_{12} & \cdots & \color{red}{c_{1i}} & \cdots & c_{1n} & d_1 \\
\vdots & \vdots & \vdots & \color{red}{\vdots} & \ddots & \vdots & \vdots \\
c_{k1} & c_{12} & \cdots & \boxed{\color{red}{c_{ki}}} & \cdots & c_{kn} & d_1 \\
\vdots & \vdots & \vdots & \color{red}{\vdots} & \ddots & \vdots & \vdots \\
c_{m1} & c_{m2} & \cdots & \color{red}{c_{1i}} & \cdots & c_{mn} & d_m
\end{array}
\quad \underset{\text{row operations}}{\overset{\longrightarrow}{\text{elementary}}} \quad
\begin{array}{cccccc|c}
x_1 & x_2 & \cdots & x_i & \cdots & x_n & \\
\hline
\tilde{c}_{11} & \tilde{c}_{12} & \cdots & \color{red}{0} & \cdots & \tilde{c}_{1n} & \tilde{d}_1 \\
\vdots & \vdots & \vdots & \color{red}{\vdots} & \ddots & \vdots & \vdots \\
\tilde{c}_{k1} & \tilde{c}_{12} & \cdots & \boxed{\color{red}{1}} & \cdots & \tilde{c}_{kn} & \tilde{d}_k \\
\vdots & \vdots & \vdots & \color{red}{\vdots} & \ddots & \vdots & \vdots \\
\tilde{c}_{m1} & \tilde{c}_{m2} & \cdots & \color{red}{0} & \cdots & \tilde{c}_{mn} & \tilde{d}_m
\end{array}
$$

The pivot around c_{ki} can be formalized as

$$
\begin{cases}
\tilde{c}_{k,\cdot} = \dfrac{1}{c_{ki}} c_{k,\cdot} \\
\tilde{c}_{j,\cdot} = c_{j,\cdot} - \dfrac{c_{ji}}{c_{ki}} c_{k,\cdot}, \quad j \neq k\,.
\end{cases}
$$

Simplex tableau of a LP

So far, we have written down simplex tableaux only for the constraint systems. Now, in addition to the constraint system

$$Cx = d, \quad x \geq 0\,,$$

we consider the objective

$$z = f_1x_1 + \cdots + f_nx_n + f_0 \quad \iff \quad f_1x_1 + \cdots + f_nx_n - z = -f_0\,.$$

We shall put it at the top of the simplex tableau.

Simplex tableau of a LP

One may view $-z$ as a variable and require that require that its row is always $(1, 0, \ldots, 0)$ so that it is forced to be basic. Then the full simplex tableau becomes

$$
\begin{array}{|ccccc|c|}
\multicolumn{1}{c}{x_1} & x_2 & \cdots & x_n & \multicolumn{1}{c}{-z} & \multicolumn{1}{c}{} \\
\hline
f_1 & f_2 & \cdots & f_n & 1 & -f_0 \\
\hline
c_{11} & c_{12} & \cdots & c_{1n} & 0 & d_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{mn} & 0 & d_m \\
\hline
\end{array}
$$

Simplex tableau of a LP

Since we do not pivot on the objective row and do not touch the column of $-z$, we may neglect this special column and only treat the following simplified tableau

$$
\begin{array}{cccc|c}
x_1 & x_2 & \cdots & x_n & \\
\hline
f_1 & f_2 & \cdots & f_n & -f_0 \\
\hline
c_{11} & c_{12} & \cdots & c_{1n} & d_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{mn} & d_m \\
\hline
\end{array}
$$

Simplex-canonical form

A tableau

$$
\begin{array}{cccc|c}
x_1 & x_2 & \cdots & x_n & \\
\hline
f_1 & f_2 & \cdots & f_n & -f_0 \\
\hline
c_{11} & c_{12} & \cdots & c_{1n} & d_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{mn} & d_m
\end{array} \tag{5}
$$

is said to be in a **simplex-canonical form** if

- there are m variable columns (called basic columns) whose lower left sections are arranged into an identity matrix,
- each basic column has only one nonzero entry, and
- all d_k's are non-negative.

The variables assigned to those columns are identified as **basic** and the rest as **nonbasic**.

One advantage of such a form is that we could obtain a basic solution by setting all the basic variables to the corresponding RHS and the nonbasic ones to 0.

Where to pivot?

Let us demonstrate this through an example. Suppose that we are at the following state of a simplex tableau

x_1	x_2	x_3	x_4	x_5	
1	0	0	-1	0	-2
1	1	0	2	0	1
2	0	0	1	1	2
0	0	1	-2	0	1

We may observe that the basic variables are x_2, x_5, x_3 (in the order of an identity matrix construction) and our basic feasible point is now $\hat{x} = (0, 1, 1, 0, 2)$.

Where to pivot?

x_1	x_2	x_3	x_4	x_5	
1	0	0	-1	0	-2
1	1	0	2	0	1
2	0	0	1	1	2
0	0	1	-2	0	1

The following observations are made.

- Looking at the objective row, we may see that the coefficient of $x_4 < 0$.
- This indicates that the increase of x_4 from 0 is benefitial because it lowers the objective value.

Our plan now is to let x_4 enters the basic variables and choose one of the current basic ones ot leave.

Keeping x_1 nonbasic (hence it is 0) and allowing only the possible vanishing of basic variables, we need to obey all of the following to maintain feasibility:

$$x_2 + 2x_4 = 1, \qquad x_4 + x_5 = 2, \qquad x_3 - 2x_4 = 1\,.$$

This shows that x_4 could not be increased beyond $x_4 = \frac{1}{2}$ otherwise either the first equation or the non-negativity of x_1 is violated.

Hence, we pivot at the first row of the constraints. As a result, the nonbasic variable x_4 swaps in place of the basic variable x_2.

Where to pivot?

x_1	x_2	x_3	x_4	x_5	
1	0	0	-1	0	-2
1	1	0	(2)	0	1
2	0	0	1	1	2
0	0	1	-2	0	1

$\underset{\text{pivot}}{\rightarrow}$

x_1	x_2	x_3	x_4	x_5	
$3/2$	$1/2$	0	0	0	$-3/2$
$1/2$	$1/2$	0	1	0	$1/2$
$3/2$	$-1/2$	0	0	1	$3/2$
1	1	1	0	0	2

Now the new basic feasible point is $\tilde{x} = \left(0, 0, 2, \frac{1}{2}, \frac{3}{2}\right)$.
Meanwhile, the above left tableau says $-z = -2$ (or $z = 2$) and the right one says $-z = -\frac{3}{2}$ (or $z = \frac{3}{2}$) — decreasing the objective value.

Coming back to the pivoting rule, we choose the row k^* in such a way that

$$k^* \in \underset{1 \leq k \leq m}{\arg\min} \left\{ \frac{d_k}{c_{ki}} \;\middle|\; d_k > 0,\ c_{ki} > 0 \right\} .$$

Simplex algorithm (Phase 2 algorithm)

Given a current state of a simplex tableau in simplex-canonical form (5).

Step 1. Pick an **entering variable** by looking at the column i^* such that $f_{i^*} < 0$ and go to Step 2.

- If $f_i \geq 0$ for all columns i, the current basic feasible point is optimal and the algorithm stops.

Step 2. Pick a **leaving variable** by choosing $k^* \in \arg\min_{1 \leq k \leq m} \left\{ \frac{d_k}{c_{ki^*}} \;\middle|\; d_k > 0,\ c_{ki^*} > 0 \right\}$ and go to Step 3.

- If $c_{ki} \leq 0$ for all $1 \leq k \leq m$, then the problem is unbounded. There is no optimal solution and the algorithm stops.

Step 3. Pivot at $c_{k^* i^*}$ and calculate the new basic feasible point. Then go to Step 1.

An example

We want to solve the following LP

$$\begin{cases} \max & 3x_1 + 2x_2 \\ \text{s.t.} & 2x_1 + x_2 \leq 18 \\ & 2x_1 + 3x_2 \leq 42 \\ & 3x_1 + x_2 \leq 24 \\ & x_1, x_2 \geq 0 \end{cases} \quad \xrightarrow[\text{canonical form}]{\text{converted into a}} \quad \begin{cases} \min & -3x_1 - 2x_2 \\ \text{s.t.} & 2x_1 + x_2 + s_1 = 18 \\ & 2x_1 + 3x_2 + s_2 = 42 \\ & 3x_1 + x_2 + s_3 = 24 \\ & x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases}$$

This corresponding tableau reads

$$\begin{array}{rrrrr|r} x_1 & x_2 & s_1 & s_2 & s_3 & \\ \hline -3 & -2 & 0 & 0 & 0 & 0 \\ \hline 2 & 1 & 1 & 0 & 0 & 18 \\ 2 & 3 & 0 & 1 & 0 & 42 \\ 3 & 1 & 0 & 0 & 1 & 24 \end{array} \tag{6}$$

Note that this is already in a simplex-canonical form.

An example (Cont.)

Let's start the simplex algorithm from this initial ready-to-use tableau (6).

x_1	x_2	s_1	s_2	s_3	
-3	-2	0	0	0	0
2	1	1	0	0	18
2	3	0	1	0	42
(3)	1	0	0	1	24

$\underset{\text{pivot}}{\longrightarrow}$

x_1	x_2	s_1	s_2	s_3	
0	-1	0	0	1	24
0	$1/3$	1	0	$-2/3$	2
0	$7/3$	0	1	$-2/3$	26
1	$1/3$	0	0	$1/3$	8

x_1	x_2	s_1	s_2	s_3	
0	-1	0	0	1	24
0	($1/3$)	1	0	$-2/3$	2
0	$7/3$	0	1	$-2/3$	26
1	$1/3$	0	0	$1/3$	8

$\underset{\text{pivot}}{\longrightarrow}$

x_1	x_2	s_1	s_2	s_3	
0	0	3	0	-1	30
0	1	3	0	-2	6
0	0	-7	1	4	12
1	0	-1	0	1	6

An example (Cont.)

x_1	x_2	s_1	s_2	s_3	
0	0	3	0	-1	30
0	1	3	0	-2	6
0	0	-7	1	(4)	12
1	0	-1	0	1	6

$\xrightarrow{\text{pivot}}$

x_1	x_2	s_1	s_2	s_3	
0	0	1.25	0.25	0	33
0	1	-0.5	0.5	0	12
0	0	-1.75	0.25	1	3
1	0	0.75	-0.25	0	3

In this last tableau, none of the coefficients in the objective row is negative. This indicates that we have arrived at an optimal state.

One could finally retrieve a basic optimal solution

$$(x_1^*, x_2^*, s_1^*, s_2^*, s_3^*) = (3, 12, 0, 0, 3)\,.$$

from this tableau.

Discarding all the slack variables, we obtain

$$x^* = (3, 12)\,.$$

Section 4

Phase 0 and Phase 1 algorithms

Phase 0 algorithm

Starting with a real LP in question, the initial simplex tableau might not be in the simplex-canonical form. Here, the Phase 0 algorithm is the first step in bringing the initial tableau into the correct form.

In fact, this is just the process of reducing the tableau into its reduced row echelon form (rref) and discarding the zero rows.

The Phase 0 algorithm essentially transforms a tableau into a canonical form but not necessarily a simplex-canonical form since the condition $d \geq 0$ might fail.

Phase 1 algorithms

To fix the potential problem from the Phase 0 algorithm that the resulting tableau has $d \not\geq 0$, we introduce the Phase 1 algorithm as follows.

Step 1. If $d < 0$, pivot the last row at a negative entry and go to Step 2.

Step 2. Suppose that there are T rows, namely $k_1, \ldots, k_T$, with positive rhs then go to Step 3.

Step 3. Swap all those T rows to the bottom of the tableau and go to Step 4.

Step 4. Regard the row $m - T$ as if it is the objective row. Take the T rows below it as the constraints and follow the simplex algorithm (Phase 2 algorithm) while the pivot is performed on the complete tableau. One stops once the rhs of this surrogate objective is positive and go to Step 5.

Step 5. If $T + 1 < m$, replace T with $T + 1$ and go to Step 4. Otherwise, the algorithm terminates and one obtains a tableau in a simplex-canonical form.

Practical implementation of the simplex algorithm

When one comes up with a LP and would like to solve it with the simplex algorithm, the full steps would be to start with the Phase 0 and Phase 1 algorithms to put the tableau into the simplex-canonical form. Then the actual simplex (Phase 2) algorithm is used to solve for an optimal solution.

Another note to take is that the simplex algorithm is fundamentally a sequence of pivots. This means the simplex algorithm favors more variables rather than many constraints.

An example

Consider the following LP

$$\begin{cases} \min & 2x_1 + 3x_2 \\ \text{s.t.} & x_1 + 2x_2 \geq 3 \\ & 2x_1 + x_2 \geq 3 \\ & x_1, x_2 \geq 0 \end{cases} \quad \underset{\text{converted into a canonical form}}{\longrightarrow} \quad \begin{cases} \min & 2x_1 + 3x_2 \\ \text{s.t.} & -x_1 - 2x_2 + s_1 = -3 \\ & -2x_1 - x_2 + s_2 = -3 \\ & x_1, x_2, s_1, s_2 \geq 0 \end{cases}$$

The problem then results in the following tableau

x_1	x_2	s_1	s_2	
2	3	0	0	0
-1	-2	1	0	-3
-2	-1	0	1	-3

$\longrightarrow$ Phase 0 and Phase 1 algorithms

x_1	x_2	s_1	s_2	
0	0	$4/3$	$1/3$	-5
1	0	$-5/3$	$-2/3$	1
0	1	$-2/3$	$1/3$	1

After applying the Phase 0 and Phase 1 algorithms, we end up with a simplex-canonical form. Note that we do not need the Phase 2 algorithm at all because the tableau is already in an optimal form. The solution is thus $x^* = (1, 1)$.

That's it!

Key takeaways.

- The canonical and simplex-canonical forms
- The implementation sequence of the Phase 0, Phase 1 and Phase 2 algorithms.
- The number of variable preference over the number of constraints.

Thank you.