Optimization

Lecture 3: Geometry of optimization and convexity

Parin Chaipunya
KMUTT
└ Mathematics @ Faculty of Science
└ The Joint Graduate School of Energy and Environments

Areas of research:

- Multi-agent optimization: Bilevel programs, Game theory
- Optimization modeling: mainly focused on energy and environmental applications

parin.cha@kmutt.ac.th
parinchaipunya.com
github.com/parinchaipunya

Last update: Januaray 2026

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Geometry of an optimization problem

Subsection 1

Level sets

Level sets

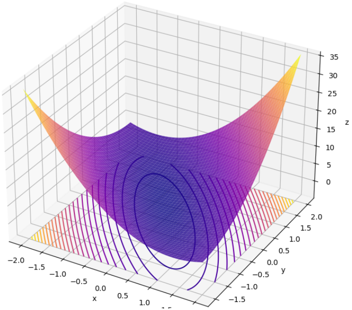
To understand the structure of an optimization problem, it is usually useful to start with the level sets (contour lines) over the decision space $\mathcal{X} = \mathbb{R}^2$.

Level sets are useful to understand the objective function as well as the equality constraints.

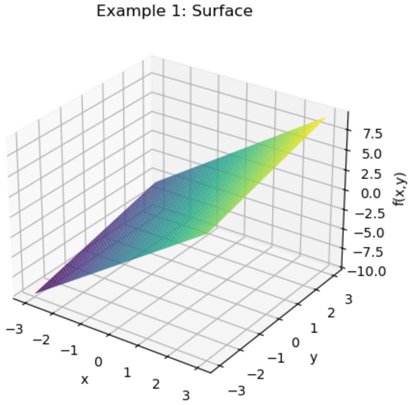
Let $f : \mathcal{X} \to \mathbb{R}$ be a function and let $\alpha \in \mathbb{R}$.

- The **level set** (or **contour**) of f at the value α is the set

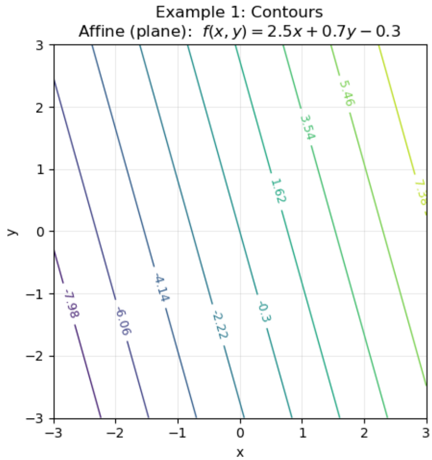
$$L_f(\alpha) = \{x \in \mathcal{X} \mid f(x) = \alpha\}.$$



Level sets

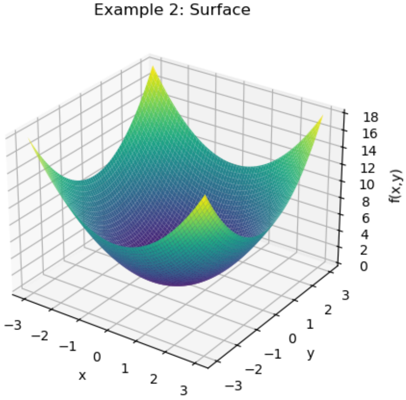


Example 1: Surface
f(x,y)
x
y

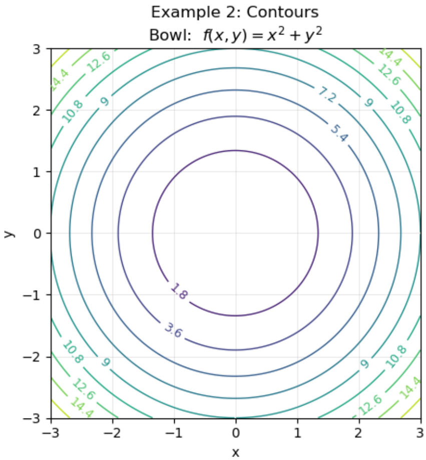


Example 1: Contours
Affine (plane): $f(x, y) = 2.5x + 0.7y - 0.3$
-7.98
-6.06
-4.14
-2.22
-0.3
1.62
3.54
5.46
x
y

Level sets

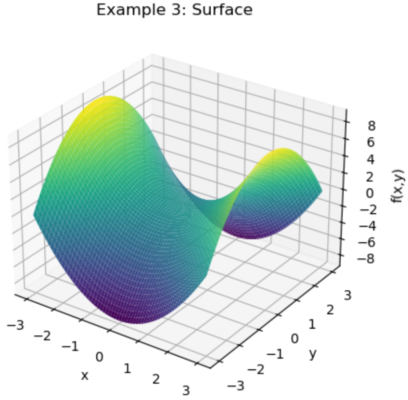


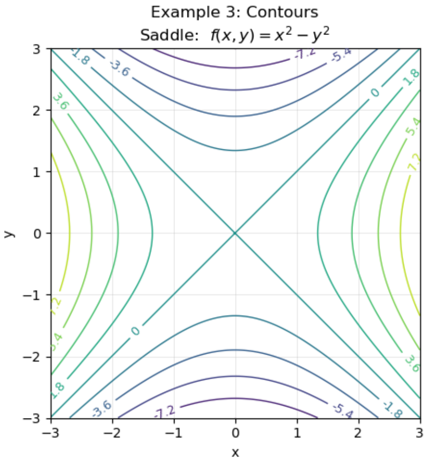
Example 2: Surface
18
16
14
12
10
8
6
4
2
0
f(x,y)
−3
−2
−1
0
1
2
3
x
−3
−2
−1
0
1
2
3
y



Example 2: Contours
Bowl: $f(x, y) = x^2 + y^2$
1.8
3.6
5.4
7.2
9
10.8
12.6
14.4
3
2
1
0
−1
−2
−3
y
−3
−2
−1
0
1
2
3
x

Level sets


Example 3: Surface
f(x,y)
x
y


Example 3: Contours
Saddle: $f(x, y) = x^2 - y^2$
x
y

Subsection 2

Sublevel sets

Sublevel sets

The sublevel sets are useful to visualize equality constraints.

Let $f : \mathcal{X} \to \mathbb{R}$ be a function and let $\alpha \in \mathbb{R}$.

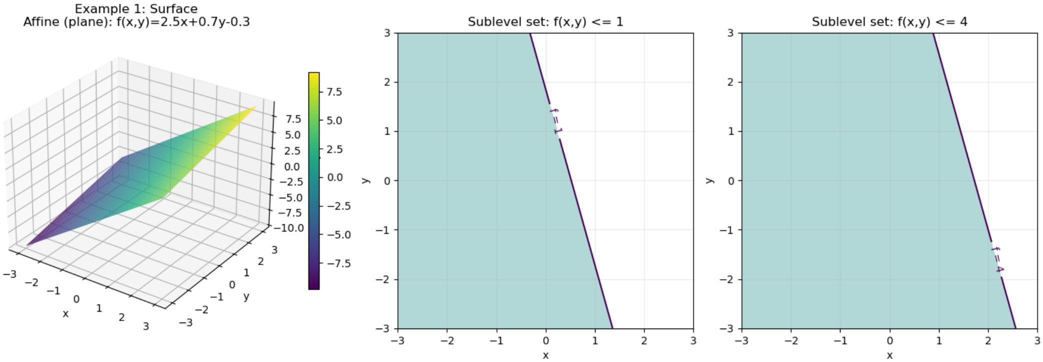
- The **sublevel set** of f at the value α is the set

$$S_f(\alpha) = \{x \in \mathcal{X} \mid f(x) \leq \alpha\}.$$

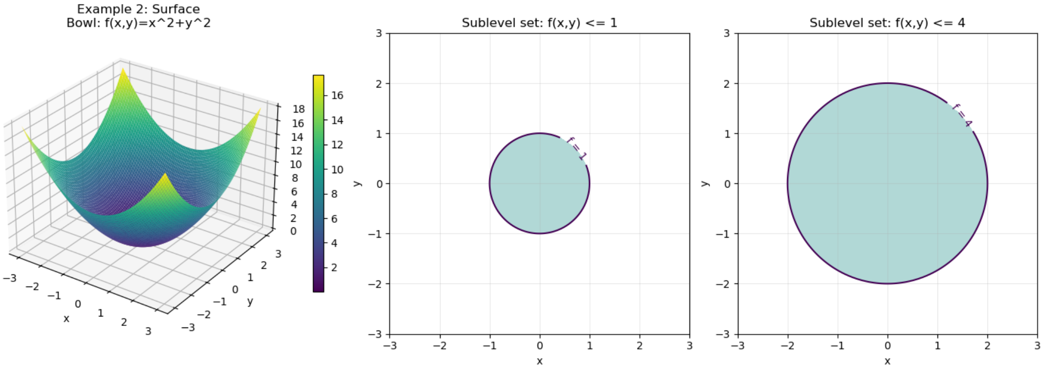
One may easily observe that

$$\alpha \leq \beta \implies S_f(\alpha) \subset S_f(\beta).$$

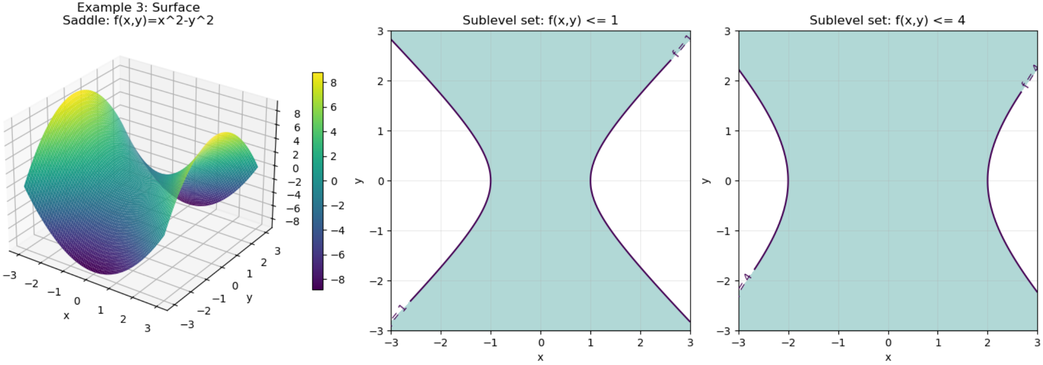
Sublevel sets



Sublevel sets



Sublevel sets



Subsection 3

Visualizing an optimization problem

A toy example

$$\begin{cases} \min & x_1 - x_2 \\ \text{s.t.} & x_1 + x_2 \leq 1 \\ & x_1 - 2x_2 = 0 \\ & x_1, x_2 \geq 0 \end{cases}$$

One may observe from this visualization that a minimizer is $\bar{x} = (0, 0)$.

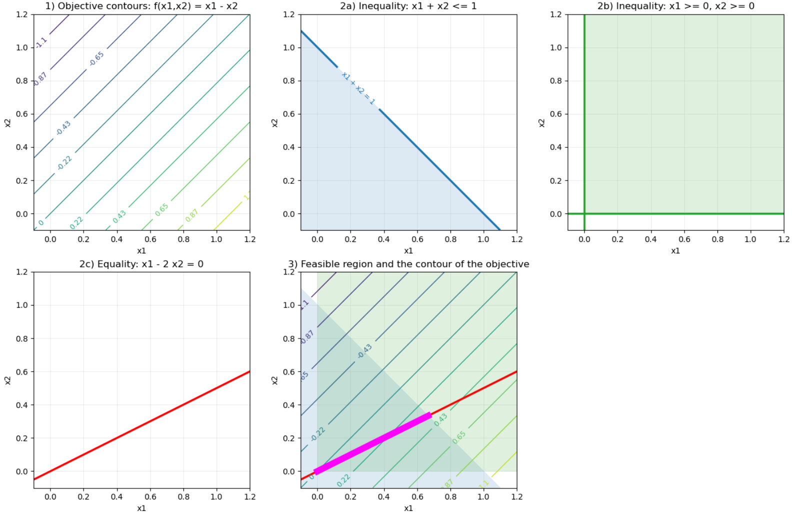


Figure: The feasible region in “magenta”.

Section 2

Convexity

Subsection 1

Convex sets and functions

Convex sets

A set $C \subset \mathbb{R}^n$ is **convex** if we have

$$(1-t)u + tv \in C \quad \text{for any } t \in [0,1] \text{ and any } u, v \in C.$$

Examples

- A line segment is convex.
- A hyperplane (a set defined by linear equalities) is convex.
- A half space (a set defined by linear inequalities) is convex.
- An intersection of convex sets is convex.
- A polyhedron (an intersection of half spaces) is convex.

Convex functions

A function $J : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}$, defined on a convex set $\mathcal{X}$, is

- **convex** if
  for any $u, v \in \mathcal{X}$ and any $t \in [0, 1]$,

$$J((1-t)u + tv) \leq (1-t)J(u) + tJ(v),$$

- **strictly convex** if
  for any $u, v \in \mathcal{X}$, $u \neq v$ and any $t \in (0, 1)$,

$$J((1-t)u + tv) < (1-t)J(u) + tJ(v),$$

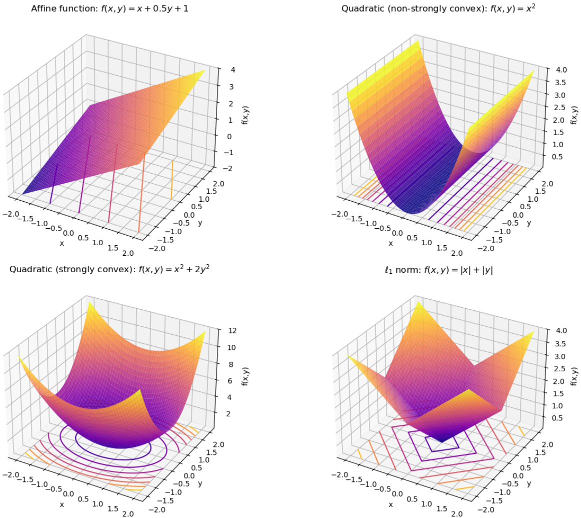
- **strongly convex** (with modulus $a > 0$) if
  for any $u, v \in \mathcal{X}$ and any $t \in [0, 1]$,

$$J((1-t)u + tv) \leq (1-t)J(u) + tJ(v) - \frac{a}{2}t(1-t)\|u - v\|^2.$$

Fact

A strongly convex function is strictly convex, and a strictly convex function is convex.

Convex functions



Convex functions

Theorem

Consider a function $f : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}$. Then

- f is convex $\iff$ $\operatorname{epi} f = \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R} \mid \lambda \geq f(x)\}$ is a convex set,
- if f is convex, then its sublevel sets are convex.

Examples

- Affine and linear functions are convex.
- A quadratic function $f(x) = \frac{1}{2}x^t Qx + a^t x + a_0$ is convex $\iff$ Q is PSD.
- A quadratic function is strictly convex $\iff$ it is strongly convex $\iff$ Q is PD.
- The function $f(x_1, x_2) = x_1 x_2$ is an example of a quadratic function that is not convex.
- If f is convex and $\alpha > 0$, then αf is convex.
- Any sum of convex functions is convex.
- A maximum of convex functions* is again convex.

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* $f(x) = \max\{f_i(x) \mid i = 1, \ldots, m\}$, where f_i's are convex.

Subsection 2

Hessian characterizations

Identifying convexities

It is not always easy to verify convexity of a function using its definition. When a function f has a Hessian $\nabla^2 f$ everywhere, then we can use them to identify the convexity of f.

Theorem

If $f : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}$ is a function whose Hessian $\nabla^2 f(x)$ exists at every $x \in \mathcal{X}$, then

- f is convex $\iff$ $\nabla^2 f(x)$ is PSD at every $x \in \mathcal{X}$.
- If $\nabla^2 f(x)$ is PD at every $x \in \mathcal{X}$, then f is strictly convex. (The converse is not true; see the next slide.)
- f is strongly convex $\iff$ there exists $M > 0$ in which the eigenvalues of $\nabla^2 f(x)$ are $\geq M$ at every $x \in \mathcal{X}$.

Identifying convexities

One-dimensional examples

- A quadratic function $f(x) = ax^2 + bx + c$, with $a > 0$ is strongly convex, since $f''(x) = a > 0$ for all $x \in \mathbb{R}$.
- An exponential function $f(x) = a^x$ with $a > 0$ is strictly convex, but not strongly convex, since $f''(x) = [\ln(a)]^2 a^x > 0$ for all $x \in \mathbb{R}$ but $f'(x) \to 0$ as $x \to -\infty$.
- The function $f(x) = x^4$ is strictly convex, but $f''(0) = 0$. This also confirms that f is not strongly convex.
- The function $f : \mathbb{R}^+ \to \mathbb{R}$, $f(x) = -\ln(x)$ has a second derivative derivative $f''(x) = \frac{1}{x^2} > 0$ at all $x \in \mathcal{X} = \mathbb{R}^+$. This means f is strictly convex.

Identifying convexities

Multi-dimensional examples

- The quadratic function $f(x) = \frac{1}{2}x^tQx + a^tx + a_0$ has a Hessian $\nabla^2 f(x) = Q$ at every x. Therefore one may observe its convexity, strict convexity, strong convexity directly from the eigenvalues of Q.
- The squared norm $f(x) = \|x\|^2 = x^tIx$ is strongly convex since $\nabla^2 f(x) = 2I$ at every x.
- The function $f(x) = \|Ax - b\|^2$ is convex since its Hessian is $\nabla^2 f(x) = 2A^tA$ at every x.

Section 3

Classes of optimization problems

Linear programs (LPs)

An optimization problem is identified as a **linear program** (or **LP**) if

- the objective function J is affine,
- the inequality constraint functions g_j's are affine,
- the equality constraint functions h_k's are affine.

This reads

$$\begin{cases} \min & f_1 x_1 + \cdots + f_n x_n + f_0 = f^t x + f_0 \\ \text{s.t.} & a_{j1} x_1 + \cdots + a_{jn} x_n \leq b_j \qquad \forall j = 1, \ldots, r \\ & c_{k1} x_{k1} + \cdots + c_{kn} x_n = d_k \qquad \forall k = 1, \ldots, \ell \,. \end{cases}$$

Quadratic programs (QPs)

An optimization problem is identified as a **quadratic program** (or **QP**) if

- the objective function J is quadratic,
- the inequality constraint functions g_j's are affine,
- the equality constraint functions h_k's are affine.

This reads

$$\begin{cases} \min & \frac{1}{2}x^t Qx + f^t x + f_0 \\ \text{s.t.} & a_{j1}x_1 + \cdots + a_{jn}x_n \leq b_j \qquad \forall j = 1, \ldots, r \\ & c_{k1}x_{k1} + \cdots + c_{kn}x_n = d_k \qquad \forall k = 1, \ldots, \ell \,. \end{cases}$$

Convex programs (CPs)

An optimization problem is identified as a **convex program** (or **CP**) if
- the objective function J is convex,
- the inequality constraint functions g_j's are convex,
- the equality constraint functions h_k's are affine.

Observation
- A linear program is a convex program.
- A quadratic program is a convex program if Q is PSD.

Safe zones

In most part of mathematics, we draw a line between $\underbrace{\text{linear}}_{\text{easy}}$ and $\underbrace{\text{nonlinear}}_{\text{difficult}}$ problems.

In optimization, we draw a line between $\underbrace{\text{convex}}_{\text{easy}}$ and $\underbrace{\text{nonconvex}}_{\text{difficult}}$ problems.

Among the convex problems, the LPs and convex QPs are extremely tractable and they are solvable with off-the-shelf solvers.

That's it!

Key takeaways.

- Visualization of an optimization problem.
- Convexity of sets and functions.
- Classes of optimization problems (LPs, QPs, CPs).

Thank you.