Optimization

Lecture 1: First glances

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Areas of research:

- Multi-agent optimization: Bilevel programs, Game theory
- Optimization modeling: mainly focused on energy and environmental applications

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General ideas about optimization

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An optimization dictionary

What is optimization ?

Optimization
(n.) A process of making something as good as it can be.

Improvement
(n.) The act of making something better.

Mathematical optimization
(n.) Using and/or creating mathematical tools to do optimization.

Where is optimization in the map of data analytics?

The four types of data analytics:

- **Descriptive:** What happened in the past ?
  Classical statistical tools, Data visualization, Clustering, etc.
- **Diagnostic:** Why something happened in the past ?
  Data discovery, Data mining, Root cause analysis, etc.
- **Predictive:** What is likely to happen in the future ?
  Regression, Classification, etc.
- **Prescriptive:** How to make something happen ?
  Optimization, Operations reseach, Reinforcement learning (repeated decision), etc.

!! One should notice that Optimization remains the main tool when it comes to prescriptive analytics.

!! Machine learning tools are largely predictive and descriptive in nature.

!! ML tools are also useful to incorporate with Optimization.

Ingredient dictionary of an optimization problem

Ingredient	**Intuition**	**Expression**
Decision variable	What you control.	$x \in \mathcal{X}$
Optimization space	The scope of what you control.	$\mathcal{X}$
Objective function	The quantitative evaluation (outcome) of the decision.	$J : \mathcal{X} \to \mathbb{R}$
Constraints	Limitations, represented as a set of feasible decisions.	$\mathcal{X}^{\text{feas}} \subset \mathcal{X}$

Decision variable

It is typical that $\mathcal{X} = \mathbb{R}^n$ (n real variables), which means x is actually a vector

$$x = (x_1, \ldots, x_n).$$

Optimization sense

An objective function J could represent different quantities, *e.g.*

- **badness:** cost, disutility, etc. (The lower, the better.)
- **goodness:** profit, utility, etc. (The higher, the better.)

If J displays the badness, then we are looking at a **minimization problem**

$$\min_{x \in \mathcal{X}^{\text{feas}}} J(x).$$

If J displays the goodness, then we are looking at a **maximization problem**

$$\max_{x \in \mathcal{X}^{\text{feas}}} J(x).$$

Useful notations

- $\arg\min_{x \in \mathcal{X}^{\text{feas}}} J(x) = \{\text{Admissible decisions that minimizes } J(x)\}$
- $\arg\max_{x \in \mathcal{X}^{\text{feas}}} J(x) = \{\text{Admissible decisions that maximizes } J(x)\}$

Optimization sense

Here and forward, it is conventional that the min, max, arg min and arg max are taken over $\mathcal{X}^{\text{feas}}$, unless stated otherwise.

Observations

- $\min J(x) = -\max\,[-J(x)]$
- $\max J(x) = -\min\,[-J(x)]$
- $\arg\min J(x) = \arg\max\,[-J(x)]$ (arg min is obtained from arg max of negative objective.)
- $\arg\max J(x) = \arg\min\,[-J(x)]$ (arg max is obtained from arg min of negative objective.)

Useful notes

For $\lambda > 0$ and $\beta \in \mathbb{R}$, we have the following.

- $\min[\lambda J(x)] = \lambda \min J(x)$
- $\min[J(x) + \beta] = [\min J(x)] + \beta$
- $\arg\min[\lambda J(x)] = \arg\min J(x)$ (The positive scaling does not affect arg min.)
- $\arg\min[J(x) + \beta] = \arg\min J(x)$ (Shifting up or down does not affect arg min.)

That being said, we shall **focus mainly on the theory of minimization**.

Constraints

Usually the constraint set $\mathcal{X}^{\text{feas}}$ is **explicit**, which means it is defined using equations and inequalities

$$\begin{aligned}\mathcal{X}^{\text{feas}} &= \left\{ x \in \mathcal{X} \;\middle|\; \begin{array}{ll} g_j(x) \leq 0 & j = 1, \dots, r \\ h_k(x) = 0 & k = 1, \dots, \ell. \end{array} \right\} \\ &= \{\text{Decisions satisfying certain equations and inequalities.}\}.\end{aligned}$$

Additionally, we may require some slices of x to be integers. In this case, we have $\mathcal{X} = \mathbb{R}^n$ and

$$\mathcal{X}^{\text{feas}} = \left\{ x \in \mathcal{X} \;\middle|\; \begin{array}{ll} g_j(x) \leq 0 & j = 1, \dots, r \\ h_k(x) = 0 & k = 1, \dots, \ell \\ x_r \in \mathbb{Z} & r \in K_0 \end{array} \right\}.$$

Here, $K_0 \subset \{1, \dots, n\}$ represents the set of indices of x_i's that are required to be integers.

General framework

An **optimization problem** (or a **mathematical program**, or a **mathematical programming problem**) takes the following form.

$$
\begin{cases}
\overbrace{\min / \max}^{\text{optimization sense}} \quad \overbrace{J(x)}^{\text{objective function}} \\
\text{s.t.} \quad \left.\begin{array}{l} g_j(x) \leq 0 \quad j = 1, \ldots, r \\ h_k(x) = 0 \quad k = 1, \ldots, \ell \\ \underbrace{x \in C}_{\text{possibly further limitations}} \end{array}\right\} \text{constraints}
\end{cases}
$$

Constrained and unconstrained problems

An optimization problem is called **unconstrained** if $\mathcal{X}^{\text{feas}} = \mathcal{X}$, *i.e.* all decisions are feasible.

Otherwise, we have $\mathcal{X}^{\text{feas}} \subsetneq \mathcal{X}$ and the problem is said to be **constrained**.

Unconstrained optimization problems are usually much easier, but most of the practical problems are constrained.

Subsection 2

Scope of this course

Scope of this course

This course covers the following topics:

- General theory of optimization (Unconstrained and then Constrained).
- Gradient-based algorithms for unconstrained and constrained optimization.
- Classes of optimization problems and their solution techniques.
- Applications in Machine learning and Operations research.

Section 2

Some examples

Subsection 1

Fencing

Fencing

Once upon a time, there is Mr.Ben.

- His uncle wants to give a part of his land to him.
- The uncle offers Mr.Ben a rope of length 200m and agrees to give him the portion of land that he could fence with the rope.
- If Mr.Ben wants a rectangular piece of land, how should he fence the land so that the area enclosed by the rope is maximized?

Fencing

Decision variables
Mr.Ben decides the dimension (width and length) of the area to be enclosed:

$$x = (\underbrace{x_1}_{\text{width (m)}}, \underbrace{x_2}_{\text{length (m)}}) \in \mathcal{X} = \mathbb{R}^2.$$

Objective function.
Mr.Ben evaluates his decision x with the resulting enclosed area, which is

$$J(x) = J(x_1, x_2) = x_1 x_2 \,.$$

This amount is to be maximized.
Constraints.
There are a few constraints in this problem.

- ◦ Each variable is non-negative: $x_1 \geq 0$ and $x_2 \geq 0$.
- ◦ The dimension is compatible with the rope's length: $2x_1 + 2x_2 = 200$.

Fencing

Putting everything together, we obtain the following optimization problem.

$$\begin{cases} \max & x_1 x_2 \\ \text{s.t.} & 2x_1 + 2x_2 = 200 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \,. \end{cases}$$

Subsection 2

Folding a box

Folding a box

- Mr.Anant has a rectangular cardboard of dimension $30 \times 50\,\text{cm}^2$.
- He wants to trim its corners and fold it into a rectangular box.
- How to formulate an optimization problem to determing the largest volume that Mr.Anant could achieve by making such a box?

Folding a box

Decision variables
The cut length $x \in \mathcal{X} = \mathbb{R}$ (cm) at each corner.
Objective function.
Mr.Ben evaluates his decision x with the resulting volume.

$$J(x) = x(30 - 2x)(50 - 2x)\,.$$

This amount is to be maximized.
Constraints.
There are a few constraints in this problem.
- Non-negativity: $x \geq 0$
- Maximum cut length: $x \leq 30/2$

Folding a box

Putting every ingredients together, we have the following optimization problem.

$$\begin{cases} \max & x(30-2x)(50-2x) \\ \text{s.t.} & x \geq 0 \\ & x \leq 15\,. \end{cases}$$

Subsection 3

Soda can design

Soda can design

- A soda company would is designing a cylindrical can that could hold 300 ml of liquid.
- How to obtain the design that minimizes the material used to produce each can?

Soda can design

Decision variables
The can's dimension

$$x = (\underbrace{r}_{\text{radius (cm)}}, \underbrace{h}_{\text{height (cm)}}) \in \mathcal{X} = \mathbb{R}^2$$

Objective function.
The company evaluates its decision with the material used (in cm^2)

$$J(x) = J(r, h) = 2\pi r^2 + 2\pi rh\,.$$

This amount is to be minimized.
Constraints.
There are a few constraints in this problem.
- Non-negativity: $r \geq 0$, $h \geq 0$
- Volume constraint: $\pi r^2 h \geq 300$

Soda can design

Putting every ingredients together, we have the following optimization problem.

$$\begin{cases} \max & 2\pi r^2 + 2\pi rh \\ \text{s.t.} & \pi r^2 h \geq 300 \\ & r \geq 0 \\ & h \geq 0\,. \end{cases}$$

Subsection 4

A hungry wizard

A hungry wizard

Once upon a time in a fruit market...

- A hungry wizard has a magic sack that carries 1 ton of anything.
- He loves apple and guava, so he wants to fill his magic sack full with these fruits.
- So he goes to a fruit merchant and asks for a combination of apple and guava.
- A ton of apple is sold for 3 gold bars, and a ton of guava for 2 gold bar.
- The merchant has 0.8 ton of each fruit.
- How to formulate an optimization problem to help maximizing the merchant's sales?

A hungry wizard

Decision variable

In the merchant's model, he sets $\mathcal{X} = \mathbb{R}^2$ and

$$x = (\underbrace{x_1}_{\text{sales amount of apple (ton)}}, \underbrace{x_2}_{\text{sales amount of guava (ton)}}).$$

Objective function

The objective function is an evaluation of his sales performance (income), which is

$$J(x) = J(x_1, x_2) = \underbrace{3x_1}_{\text{apple sales}} + \underbrace{2x_2}_{\text{guava sales}}.$$

This amount is to be maximized.

Constraints

There are a few constraints.

- The two fruits fill the sack full: $x_1 + x_2 = 1$
- Stock limitations: $x_1 \leq 0.8,\ x_2 \leq 0.8$
- Non-negativity: $x_1 \geq 0,\ x_2 \geq 0$

A hungry wizard

Putting everything together, one arrives at the following optimization problem.

$$\begin{cases} \max & 3x_1 + 2x_2 \\ \text{s.t.} & x_1 + x_2 = 1 \\ & x_1 \leq 0.8 \\ & x_2 \leq 0.8 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{cases}$$

Subsection 5

Some other examples

What could it be ? — Manufacturing

$$\begin{cases} \text{min} & \text{Manufacturing cost} \\ \text{s.t.} & \text{Demand constraint} \\ & \text{Manufacturing limit} \\ & \text{Material limit} \end{cases}$$

What could it be ? — Resources

$$\begin{cases} \text{min} & \text{Resources used} \\ \text{s.t.} & \text{Order quantity} \\ & \text{Product quality} \\ & \text{Production formula} \end{cases}$$

What could it be ? — Delivery

$$\begin{cases} \text{min} & \text{Distance travelled} \\ \text{s.t.} & \text{Visits all customers} \\ & \text{No subtour} \end{cases} \qquad \begin{cases} \text{min} & \text{Travel time} \\ \text{s.t.} & \text{Visits all customers} \\ & \text{No subtour} \end{cases}$$

What could it be ? — Load shifting

$$\begin{cases} \text{min} & \text{Bill payment} + \text{discomfort} \\ \text{s.t.} & \text{Demand dynamics} \\ & \text{Shift limits} \\ & \text{Total energy preserved} \end{cases}$$

What could it be ? — Electricity production

$$\begin{cases} \text{min} & \text{Production cost} + \text{Startup cost} + \text{Shutdown cost} \\ \text{s.t.} & \text{Generation limits} \\ & \text{Transmission limits} \\ & \text{Demand balance} \\ & \text{Reserve margin constraint} \\ & \text{Emission limit} \\ & \text{Generator ramping constraints} \end{cases}$$

What could it be ? — Transmission expansion planning

$$\begin{cases} \min & \text{Investment cost} + \text{Operation cost} \\ \text{s.t.} & \text{Power flow balance} \\ & \text{Material constraint} \\ & \text{Budget constraint} \\ & \text{Reliability constraint} \end{cases}$$

What could it be ? — Electricity market clearing

$$\begin{cases} \max & \text{Social welfare} = \text{Consumer benefit} - \text{Generation cost} \\ \text{s.t.} & \text{Market balance} \\ & \text{Network limits} \\ & \text{Nodal pricing constraints} \end{cases}$$

What could it be ? — Storage management

$$\begin{cases} \text{min} & \text{Generation cost} + \text{Battery operation cost} \\ \text{s.t.} & \text{Power balance: generation} + \text{discharge} = \text{demand} + \text{charge} \\ & \text{Storage limits} \\ & \text{Storage behaviors} \end{cases}$$

What could it be ? — Microgrid optimization

$$\begin{cases} \min & \text{Cost of local generation} + \text{Import/export cost} \\ \text{s.t.} & \text{Power balances} \\ & \text{Renewable constraints} \end{cases}$$

Section 3

Optimality notion

Subsection 1

Local vs global optima

Local vs global optima

Definition

A decision $\bar{x}$ is

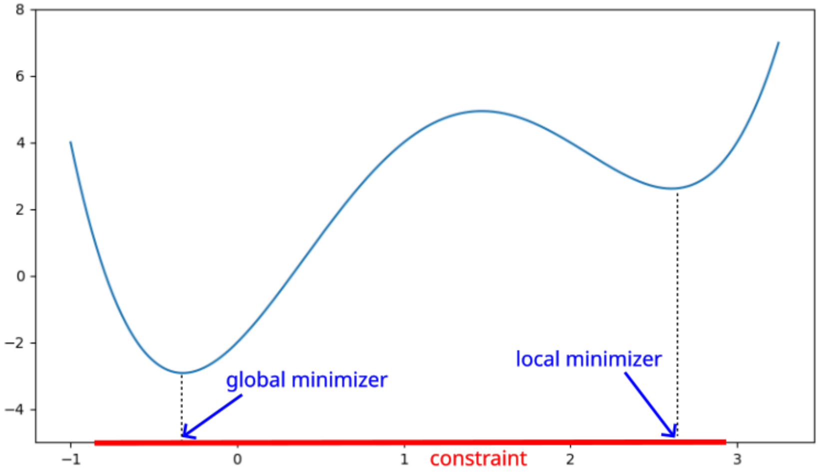
- a **global minimizer** of J if it is feasible and

$$J(\bar{x}) \leq J(x) \quad \text{for any admissible decision } x \in \mathcal{X}^{\text{feas}}.$$

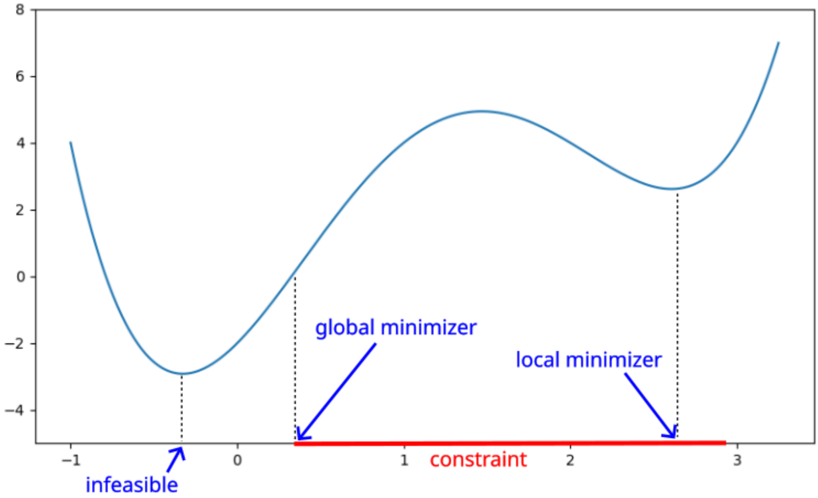
- a **local minimizer** of J if it is feasible and

$$J(\bar{x}) \leq J(x) \quad \text{for any admissible decision } x \in \mathcal{X}^{\text{feas}} \text{ near } \bar{x}.$$

Local vs global optima



Local vs global optima



Local vs global optima

It is always the best if one could find a global optimal point. However, there are some technical difficulties.

- It is difficult to identify a global solution.
- Most algorithms get stuck at local solutions.

Section 4

Existence criteria

Existence criteria

Not all problems have a (global) solution. We could think of some simple examples.

- The function $f(x) = e^x$ has a lower bound but no minimizer.
- Any linear function $f(x) = ax$ with $a \neq 0$ does not have a lower bound, it has no minimizer.

To guarantee an optimal point, we rely on the following result.

- **Weierstrass theorem for constrained optimization.**
  If J is continuous and $\mathcal{X}^{\text{feas}}$ is compact, then $\arg\min_{x \in \mathcal{X}^{\text{feas}}} J(x)$ and $\arg\min_{x \in \mathcal{X}^{\text{feas}}} J(x)$ are nonempty.
- **Existence theorem for unconstrained optimization.**
  Suppose that $\mathcal{X} \subset \mathbb{R}^n$ is nonempty and open, $J : \mathcal{X} \to \mathbb{R}$ is continuous, $\lim_{\|x\| \to \infty} J(x) = \infty$ and $\lim_{x \to \partial \mathcal{X}} J(x) = \infty$. Then $\arg\min_{x \in \mathcal{X}} J(x)$ is nonempty.

That's it!

Key concept takeaways.

- Ingredients in an optimization problem.
- Local vs global optimality.
- Existence criteria.

Thank you.